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LMI tools for eventually periodic systems

Mazen Farhood, Geir E. Dullerud*

Department of Mechanical and Industrial Engineering, University of Illinois, 1206 West Green Street, MC-244, Urbana, IL 61801, USA

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Abstract

This paper is focused on the concept of an eventually periodic linear discrete-time system. We derive a necessary and sufficient analysis condition for checking open-loop stability and performance of such systems, and use this to derive exact controller synthesis conditions given eventually periodic plants. All the conditions derived are provided in terms of semi-definite programming problems. The motivation for this work is controlling nonlinear systems along prespecified trajectories, notably those which eventually settle down into periodic orbits and those with uncertain initial states.

Keywords: Time-varying systems; Orbits; LMIs

1. Introduction

In this paper, we introduce and study *eventually periodic* systems. Such systems are aperiodic for an initial amount of time, and then become periodic afterwards. Our work is motivated by the desire to use robust control methods for control of nonlinear systems along prespecified trajectories. There are two basic ways in which eventually periodic dynamics arise when linearizing systems along trajectories: (1) the system trajectory is an aperiodic maneuver joined to a subsequent periodic orbit; or (2) the initial condition of the system is uncertain. We remark that both finite horizon and periodic systems are subclasses of eventually periodic systems.

The main technical contribution of this paper is a necessary and sufficient condition for the exponential stability and contractiveness (in the ℓ_2-induced norm) of open-loop eventually periodic systems. This condition, furthermore, is given in terms of a linear matrix inequality (LMI) feasibility problem. In other words, this paper primarily provides a version of the Kalman–Yakubovich–Popov (KYP) Lemma for eventually periodic systems. Based on the aforementioned open-loop result, we derive necessary and sufficient conditions for the existence of eventually periodic controllers which both stabilize and provide performance in closed-loop control systems. It is not difficult to show (via example) that, given an eventually periodic system of a specified class to be controlled, the closed-loop performance may only be achievable by a controller *outside* that class; plainly, the controller may need to exhibit longer transient time variation than the plant.

* Corresponding author. Tel.: +1-217-265-5078; fax: +1-217-244-653.
E-mail address: dullerud@uiuc.edu (G.E. Dullerud).

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The contributions of this paper are:

- a new LMI characterization of stability and performance for discrete-time eventually periodic systems;
- precise conditions for closed-loop synthesis of eventually periodic controllers.

We note that the aforementioned LMI results are derived for the general setting of non-stationary *dimensions* of the state space systems used.

The general machinery used to obtain the results of this paper is motivated by the work in [7,11,13], combined with the time-varying system machinery developed in [5]. Also, see the closely related earlier work in [1,9,10] on nonstationary systems. The main analysis result derived in the paper is based on [6], and is proved using a self-contained matrix inequality approach. The most closely related works to the current paper are [2,16] which give very useful results on the solutions of Riccati equations. The literature in the area of time-varying systems is vast, and we refer the reader to [8] for a comprehensive list of general references.

The paper is organized as follows. After a notational section, we review some previous results on linear time-varying (LTV) systems. We then move on to a section, partitioned into three subsections, in which we derive the main analysis result of the paper. Synthesis of eventually periodic controllers for eventually periodic plants is then presented. Finally, we provide some concluding remarks.

2. Preliminaries

We now introduce our notation and gather some elementary facts. The set of real numbers and that of real $n\times m$ matrices are denoted by $\mathbb{R}$ and $\mathbb{R}^{n\times m}$, respectively. If S_i is a sequence of operators, then $\operatorname{diag}(S_i)$ denotes their block-diagonal augmentation.

Given two Hilbert spaces E and F, we denote the space of bounded linear operators mapping E to F by $\mathscr{L}(E,F)$, and shorten this to $\mathscr{L}(E)$ when E equals F. If X is in $\mathscr{L}(E,F)$, we denote the E to F induced norm of X by $\|X\|_{E\to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. When an operator $X\in\mathscr{L}(E)$ is self-adjoint, we use $X<0$ to mean it is negative definite; that is there exists a number $\alpha>0$ such that, for all nonzero $x\in E$, the inequality $\langle x,Xx\rangle < -\alpha\|x\|^2$ holds. We now state an elementary fact used in the sequel.

Proposition 1. *Suppose X and Y are self-adjoint operators on two Hilbert spaces, and W is an operator between these spaces. Then*

$$\begin{pmatrix} X & W \\ W^* & Y \end{pmatrix} < 0$$

if and only if $Y<0$ and $X-WY^{-1}W^*<0$.

This is the well-known Schur complement formula and will be referred to as such; it can be found in any introductory text on matrix or operator theory.

We will be primarily concerned with two Hilbert spaces in this paper. The first is the standard space $\mathbb{R}^n$ with the inner product given by $\langle x,y\rangle_{\mathbb{R}^n}=\sum_{t=0}^{n-1} x_t y_t = x^*y$. The second Hilbert space of interest is formed given an infinite sequence $\{\mathbb{R}^{n_t}\}$ of Hilbert spaces, and is denoted by $\ell_2(\{\mathbb{R}^{n_t}\})$. It is defined as the subspace of the Hilbert space direct sum $\bigoplus_{t=0}^{\infty}\mathbb{R}^{n_t}$ consisting of elements $(x_0,x_1,x_2,\ldots)$ which satisfy $\sum_{t=0}^{\infty}\|x_t\|^2_{\mathbb{R}^{n_t}}<\infty$. The inner product of x and y in $\ell_2(\{\mathbb{R}^{n_t}\})$ is defined by the infinite sum $\langle x,y\rangle_{\ell_2}=\sum_{t=0}^{\infty}\langle x_t,y_t\rangle_{\mathbb{R}^{n_t}}$. In the sequel, we will frequently suppress the subscript on the dimension symbol n_t and accordingly use a shorter notation for $\ell_2(\{\mathbb{R}^{n_t}\})$, namely $\ell_2(\mathbb{R}^n)$. Also, when the spatial dimensions n_t are either evident or not relevant to the discussion, we abbreviate further to ℓ_2. We will use $\|x\|$ to denote $\sqrt{\langle x,x\rangle}$, the standard norm on this space.

One of the most important operators used in the paper is the unilateral shift operator Z defined as follows:

$$
\begin{aligned}
Z\colon \ell_2(\{\mathbb{R}^{m_k}\}) &\quad\to\quad \ell_2(\{\mathbb{R}^{n_k}\}), \quad \text{where } m_k = n_{k+1} \\
(a_0, a_1, a_2, \ldots) &\quad\overset{Z}{\mapsto}\quad (0, a_0, a_1, a_2, \ldots).
\end{aligned}
$$

Following the notation and approach in [5], we make the following definition.

Definition 2. A bounded operator Q mapping $\ell_2(\{\mathbb{R}^{m_k}\})$ to $\ell_2(\{\mathbb{R}^{n_k}\})$ is *block-diagonal* if there exists a sequence of matrices Q_k in $\mathbb{R}^{n_k \times m_k}$ such that, for all w, z, if $z = Qw$, then $z_k = Q_k w_k$. Then Q has the representation $\operatorname{diag}(Q_0, Q_1, Q_2, \ldots)$.

Suppose F, G, R and S are block-diagonal operators, and let A be a *partitioned operator*, each of whose elements is a block-diagonal operator, such as

$$
A = \begin{bmatrix} F & G \\ R & S \end{bmatrix}.
$$

We now define the following notation:

$$
\left[\!\left[\begin{matrix} F & G \\ R & S \end{matrix} \right]\!\right] := \operatorname{diag}\left(\begin{bmatrix} F_0 & G_0 \\ R_0 & S_0 \end{bmatrix}, \begin{bmatrix} F_1 & G_1 \\ R_1 & S_1 \end{bmatrix}, \ldots \right),
$$

which we call the *diagonal realization* of A. Clearly for any given operator A of this particular structure, $[\![A]\!]$ is simply A with the rows and columns permuted appropriately so that

$$
\left[\!\left[\begin{matrix} F & G \\ R & S \end{matrix} \right]\!\right]_t = \begin{bmatrix} F_t & G_t \\ R_t & S_t \end{bmatrix}.
$$

Having established these definitions, we are ready to consider the main subject of this paper.

3. LTV systems and the KYP lemma

We now briefly review LTV state space systems; see [5] for an in-depth treatment of the theory. Suppose we are considering the time-varying difference equation

$$
\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad x_0 = 0, \\
y_t &= C_t x_t + D_t u_t,
\end{aligned}
$$

where A_t, B_t, C_t and D_t are bounded *real* matrix sequences, each of which having elements of possibly different dimensions. Then clearly these sequences define block-diagonal operators A, B, C and D, and therefore the above system may be written more compactly in operator form as

$$
\begin{aligned}
x &= ZAx + ZBu, \\
y &= Cx + Du,
\end{aligned} \tag{1}
$$

where Z is the shift, or delay, operator on ℓ_2. Thus, assuming the relevant inverse exists, we can write the map from u to y as

$$
u \mapsto y = C(I - ZA)^{-1} ZB + D.
$$

It is possible to show that $I - ZA$ is invertible if and only if the system $x_{t+1} = A_t x_t$ is exponentially stable. Throughout the paper we will say an open- or closed-loop LTV state space system is *stable* when its A-operator satisfies the above invertibility condition.

A very important result in systems theory is the KYP lemma. While there are many versions of this result, we are only concerned with the one that turns an ℓ_2 induced norm condition into a linear operator inequality. This version of the KYP lemma is stated as follows.

Lemma 3. *Suppose operators A, B, C, and D are block-diagonal. The following conditions are equivalent*:

(i) $\|C(I-ZA)^{-1}ZB+D\|<1$ *and* $I-ZA$ *is invertible*;
(ii) *There exists* $\bar{X}\in\mathscr{L}(\ell_2)$, *which is self-adjoint and positive definite, such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{2}$$

The proof of this lemma follows directly from [15], employing some of the notational machinery of [5]. The condition in (ii) is clearly a linear operator inequality and hence gives us a very convenient way to evaluate the ℓ_2 induced norm of the input–output mapping $u\mapsto y$. Now this condition can be further strengthened by imposing additional structure on the operator $\bar{X}$. In fact, [5] shows that a solution exists to inequality (2) if and only if a block-diagonal solution belonging to $\mathscr{X}$ exists where the set $\mathscr{X}$ consists of positive definite self-adjoint operators X of the form

$$X=\mathrm{diag}(X_0,X_1,X_2,\ldots)>0, \tag{3}$$

with the block structure being the same as that of the operator A. Also, [5] gives another result regarding the KYP lemma in the case of periodic operators. However, before stating this result, we need to introduce some definitions.

Definition 4. An operator P on ℓ_2 is said to be q-periodic if it commutes with the q-shift, that is

$$Z^qP=PZ^q.$$

Suppose that Q is a q-periodic *block-diagonal* operator, then we define the matrix $\hat{Q}$ to be the *first period truncation* of Q, namely $\hat{Q}:=\mathrm{diag}(Q_0,\ldots,Q_{q-1})$. Also, we define the cyclic shift matrix $\hat{Z}$ for $q\geqslant 2$ by

$$\hat{Z}=\begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}, \text{ such that } \hat{Z}^*\hat{Q}\hat{Z}=\begin{bmatrix} Q_1 & & & 0 \\ & \ddots & & \\ & & Q_{q-1} & \\ 0 & & & Q_0 \end{bmatrix}.$$

For $q=1$, we set $\hat{Z}=I$.

The following theorem states that, given q-periodic block-diagonal state space operators, a solution exists to inequality (2) if and only if a q-periodic solution exists.

Theorem 5. *Suppose block-diagonal operators A, B, C and D are q-periodic, and that* (2) *has a solution in $\mathscr{X}$. Then there exists a q-periodic operator $X\in\mathscr{X}$ such that its first period truncation $\hat{X}$ satisfies*

$$\begin{bmatrix} \hat{Z}\hat{A} & \hat{Z}\hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}^* \begin{bmatrix} \hat{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{Z}\hat{A} & \hat{Z}\hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} - \begin{bmatrix} \hat{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{4}$$

The proof of this theorem is presented in [5].

4. Analysis results

In this section, we will show that a similar result to Theorem 5 holds for *eventually periodic systems*. We start by stating the following definition.

Definition 6. An operator P on ℓ_2 is said to be (k,q)-eventually periodic if, for some non-negative integer k, we have

$$Z^q((Z^*)^k P Z^k) = ((Z^*)^k P Z^k) Z^q.$$

In the case where $q = 1$, P is called k-eventually time-invariant. Also, when only the period length q is relevant, we simply call P *eventually* q*-periodic*.

Namely, a (k,q)-eventually periodic operator is q-periodic after an initial transient behavior up to time k. Throughout the sequel we set $k \geqslant 0$ and $q \geqslant 1$ to be some *fixed* integers. It is worth noting that when $k = 0$, the above operator P would simply be q-periodic.

Our goal is to show that, for a (k,q)-eventually periodic system, if inequality (2) is solvable, then it has as well a (k,q)-eventually periodic solution. Proving this requires a number of steps, and we partition our effort into the subsequent three subsections.

4.1. Eventual periodicity

The first objective is establishing the following lemma.

Lemma 7. *Suppose block-diagonal operators A, B, C and D are (k,q)-eventually periodic, and that $X \in \mathscr{X}$ and satisfies (2). Then there exists an eventually q-periodic operator $\check{X} \in \mathscr{X}$ such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} \check{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} \check{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{5}$$

The lemma says that a solution exists to inequality (2) if and only if an eventually q-periodic solution exists. We emphasize that the above lemma does not specify the length of the finite horizon of the solution. However, later in this section, we are going to present a much more powerful result which will specify this length exactly. It will turn out that this length is exactly the same as that of the finite horizon of the state space operators A, B, C and D, namely k.

Proof. The linear operator inequality (2) can be equivalently written as an infinite number of LMIs, each of which corresponds to a distinct time t. In this proof, we will make use of the continuity and convexity properties of these LMIs to show that we can construct from any solution of inequality (2) an eventually q-periodic solution. The only LMIs useful to our proof will turn out to be those corresponding to instances $t \geqslant k-1$. This means that, of all the finite horizon LMIs, only the one corresponding to the last instance of the finite horizon will be used in this proof. As a result, we will assume, without loss of generality, that the finite horizon length k is simply equal to 1. Then the state space operator A will have the representation

$$A = \operatorname{diag}(A_0, A_{\text{per}}), \text{ where } A_{\text{per}} = \operatorname{diag}(\hat{A}_{\text{per}}, \hat{A}_{\text{per}}, \ldots), \tag{6}$$

$\hat{A}_{\text{per}}$ being the first period truncation of the q-periodic block-diagonal operator A_{per}. Similar representations apply for the other state space operators.

By assumption $X \in \mathscr{X}$ satisfies (2). Therefore, there exists a real number $\beta > 0$ such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < -\beta I. \tag{7}$$

The above inequality is equivalent to the following matrix and operator inequalities:

$$\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}^* \begin{bmatrix} X_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < -\beta I, \tag{8}$$

$$\begin{bmatrix} A_{\text{per}} & B_{\text{per}} \\ C_{\text{per}} & D_{\text{per}} \end{bmatrix}^* \begin{bmatrix} Z^*\bar{X}Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\text{per}} & B_{\text{per}} \\ C_{\text{per}} & D_{\text{per}} \end{bmatrix} - \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} < -\beta I, \tag{9}$$

where $\bar{X} = \text{diag}(X_1, X_2, X_3, \ldots)$. Invoking Theorem 5, we deduce that there exists a q-periodic operator $X_{\text{per}} \in \mathscr{X}$ that solves inequality (9). In other words, if (9) holds then

$$\begin{bmatrix} \hat{Z}\hat{A}_{\text{per}} & \hat{Z}\hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix}^* \begin{bmatrix} \hat{X}_{\text{per}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{Z}\hat{A}_{\text{per}} & \hat{Z}\hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix} - \begin{bmatrix} \hat{X}_{\text{per}} & 0 \\ 0 & I \end{bmatrix} < -\beta I \tag{10}$$

holds, where $\hat{X}_{\text{per}} = \text{diag}(X_{\text{per},0}, \ldots, X_{\text{per},q-1})$ is the first period truncation of X_{per}.

Now consider matrix inequality (8), and choose $\varepsilon > 0$ such that the following holds:

$$F_0^* \begin{bmatrix} \dfrac{X_1 + \varepsilon X_{\text{per},0}}{1+\varepsilon} & 0 \\ 0 & I \end{bmatrix} F_0 - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < -\beta I, \quad \text{where } F_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}. \tag{11}$$

Also note from (9) and (10) that, for all $\mu \geqslant 0$ and $i = 1, q+1, 2q+1, \ldots,$ we have

$$\begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix}^* \begin{bmatrix} \hat{Z}^* \left(\dfrac{Y_i + \mu \hat{X}_{\text{per}}}{1+\mu} \right) \hat{Z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix} - \begin{bmatrix} \dfrac{Y_i + \mu \hat{X}_{\text{per}}}{1+\mu} & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

where $Y_i = \text{diag}(X_i, \ldots, X_{i+q-1})$. With the above inequality in mind, choose $\xi > 0$ such that, for all $\mu \geqslant 0$ and $i = q, 2q, 3q, \ldots,$ the following is true:

$$F_i^* \begin{bmatrix} \dfrac{X_{i+1} + (\mu + \xi) X_{\text{per},0}}{1 + \mu + \xi} & 0 \\ 0 & I \end{bmatrix} F_i - \begin{bmatrix} \dfrac{X_i + \mu X_{\text{per},q-1}}{1+\mu} & 0 \\ 0 & I \end{bmatrix} < -\beta I.$$

Hence, inequality (7) implies that the following inequalities hold:

$$F_0^* \begin{bmatrix} \dfrac{X_1 + \varepsilon_0 X_{\mathrm{per},0}}{1+\varepsilon_0} & 0 \\ 0 & I \end{bmatrix} F_0 - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

$$F_1^* \begin{bmatrix} \dfrac{X_2 + \varepsilon_0 X_{\mathrm{per},1}}{1+\varepsilon_0} & 0 \\ 0 & I \end{bmatrix} F_1 - \begin{bmatrix} \dfrac{X_1 + \varepsilon_0 X_{\mathrm{per},0}}{1+\varepsilon_0} & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

$$F_2^* \begin{bmatrix} \dfrac{X_3 + \varepsilon_0 X_{\mathrm{per},2}}{1+\varepsilon_0} & 0 \\ 0 & I \end{bmatrix} F_2 - \begin{bmatrix} \dfrac{X_3 + \varepsilon_0 X_{\mathrm{per},1}}{1+\varepsilon_0} & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

$$\vdots$$

$$F_q^* \begin{bmatrix} \dfrac{X_{q+1} + \varepsilon_1 X_{\mathrm{per},0}}{1+\varepsilon_1} & 0 \\ 0 & I \end{bmatrix} F_q - \begin{bmatrix} \dfrac{X_q + \varepsilon_0 X_{\mathrm{per},q-1}}{1+\varepsilon_0} & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

$$\vdots$$

$$\vdots$$

$$F_{Nq+1}^* \begin{bmatrix} \dfrac{X_{Nq+2} + \varepsilon_N X_{\mathrm{per},1}}{1+\varepsilon_N} & 0 \\ 0 & I \end{bmatrix} F_{Nq+1} - \begin{bmatrix} \dfrac{X_{Nq+1} + \varepsilon_N X_{\mathrm{per},0}}{1+\varepsilon_N} & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

$$\vdots$$

$$F_{(N+1)q}^* \begin{bmatrix} \dfrac{X_{(N+1)q+1} + \varepsilon_{N+1} X_{\mathrm{per},0}}{1+\varepsilon_{N+1}} & 0 \\ 0 & I \end{bmatrix} F_{(N+1)q} - \begin{bmatrix} \dfrac{X_{(N+1)q} + \varepsilon_N X_{\mathrm{per},q-1}}{1+\varepsilon_N} & 0 \\ 0 & I \end{bmatrix} < -\beta I,$$

$$\vdots$$

with $\varepsilon_i := \varepsilon + i\xi$. Now for $i \geqslant 1$, define

$$\check{X}_i := \frac{X_i + \varepsilon_c X_{\mathrm{per},d}}{1+\varepsilon_c},$$

where $c := \mathrm{floor}((i-1)/q)$ and $d := (i-1) \bmod q$. Set $\check{X}_0 = X_0$. With this definition, the above inequality list becomes

$$F_i^* \begin{bmatrix} \check{X}_{i+1} & 0 \\ 0 & I \end{bmatrix} F_i - \begin{bmatrix} \check{X}_i & 0 \\ 0 & I \end{bmatrix} < 0 \quad \text{for all } i \geqslant 0.$$

Clearly, $\check{X}_{(N+1)q+1}$ tends to $X_{\mathrm{per},0}$ as N tends to infinity, and thus, for some sufficiently large N, we can replace $\check{X}_{(N+1)q+1}$ in the above list by $X_{\mathrm{per},0}$ to get

$$F^*_{(N+1)q}\begin{bmatrix} X_{\mathrm{per},0} & 0 \\ 0 & I \end{bmatrix} F_{(N+1)q} - \begin{bmatrix} \check{X}_{(N+1)q} & 0 \\ 0 & I \end{bmatrix} < 0.$$

Using the first $(N+1)q$ inequalities in the above list, this inequality, and (10), it is routine to see that the eventually q-periodic operator

$$\check{X} := \mathrm{diag}(\check{X}_0, \check{X}_1, \ldots, \check{X}_{(N+1)q}, X_{\mathrm{per}})$$

solves (7). □

Having proved this intermediate result, we must now build on it to achieve the stronger result sought. In order to do this, we must first develop some new tools.

4.2. Technical machinery

We now introduce some definitions and results that are essential to proving the main result of this section. To start, we denote the set of symmetric positive semi-definite matrices of dimension n by $\mathbb{P}^n$, and define the sequence of sets $\mathbb{D}_i$ for non-negative integers i by

$$\mathbb{D}_i = \{X \in \mathbb{P}^n \colon B_i^* X B_i + D_i^* D_i - I < 0\}.$$

Also, for integers $i \geqslant 0$, we define the sequence[1] of maps $\Omega_i : \mathbb{D}_i \to \mathbb{P}^n$ by

$$\Omega_i(X) = A_i^* X A_i + C_i^* C_i - (A_i^* X B_i + C_i^* D_i)(B_i^* X B_i + D_i^* D_i - I)^{-1}(B_i^* X A_i + D_i^* C_i).$$

The following proposition is extensively used in the sequel.

Proposition 8. *Suppose $X \in \mathbb{P}^n$, $i \geqslant 0$, and $Y \in \mathbb{D}_i$. If $X \leqslant Y$, then $X \in \mathbb{D}_i$ and $\Omega_i(X) \leqslant \Omega_i(Y)$.*

Proof. Obviously, from the definition of the domain $\mathbb{D}_i$, if $Y \in \mathbb{D}_i$, then the fact that $X \leqslant Y$ implies directly that $X \in \mathbb{D}_i$. Now for every $\varepsilon > 0$, the following inequality is true:

$$\Omega_i(Y) < \underbrace{\Omega_i(Y) + \varepsilon I}_{W}.$$

Applying the Schur complement to this inequality, we get

$$F_i^* \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} F_i - \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} < 0,$$

where

$$F_i := \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}.$$

[1] Notice that the iteration $\Omega_i(X_t) = X_{t+1}$ is just the discrete-time matrix Riccati equation.

Also, we have

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix},$$

which implies that

$$F_i^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} F_i - \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \leqslant F_i^* \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} F_i - \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}.$$

Hence, the left-hand side (LHS) of the above inequality is negative definite, and so applying the Schur complement to the inequality LHS < 0, we get

$$\Omega_i(X) < \Omega_i(Y) + \varepsilon I$$

for all $\varepsilon > 0$. Thus, $\Omega_i(X) \leqslant \Omega_i(Y)$. □

Another very useful proposition is the following.

Proposition 9. *Suppose that $X \in \mathscr{X}$ and satisfies (2). Then, for each $i \geqslant 0$, both $X_{i+1} \in \mathbb{D}_i$ and $\Omega_i(X_{i+1}) < X_i$ hold.*

Proof. By assumption $X \in \mathscr{X}$ and satisfies (2). Then the following holds:

$$F_i^* \begin{bmatrix} X_{i+1} & 0 \\ 0 & I \end{bmatrix} F_i - \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} < 0$$

for all integers $i \geqslant 0$. Applying the Schur complement formula to the above LMI, we get

$$\begin{aligned} &B_i^* X_{i+1} B_i + D_i^* D_i - I < 0, \\ &\Omega_i(X_{i+1}) < X_i. \end{aligned} \tag{12}$$

Note that (12), along with the fact that $X_{i+1} > 0$, clearly implies that $X_{i+1} \in \mathbb{D}_i$. □

Before proceeding, it is convenient to define a domain and a corresponding map, which are closely related to the $\mathbb{D}_i$ and Ω_i respectively. For $q \geqslant 2$, we define the domain set $\hat{\mathbb{D}}$ by

$$\hat{\mathbb{D}} = \{X \in \mathbb{D}_{k+q-1}\colon\ \Omega_i(\Omega_{i+1}(\cdots \Omega_{k+q-1}(X)\cdots)) \in \mathbb{D}_{i-1} \text{ for all } i = k+1, \dots, k+q-1\}.$$

For $q = 1$, we set $\hat{\mathbb{D}} = \mathbb{D}_{k+q-1}$. Associated with this domain is the map $\hat{\Omega}: \hat{\mathbb{D}} \to \mathbb{P}^n$ defined by

$$\hat{\Omega}(X) = \Omega_k(\Omega_{k+1}(\cdots \Omega_{k+q-1}(X)\cdots)).$$

Last, for some integer $m \geqslant 1$, we formally define $\hat{\Omega}^m(X)$ by

$$\hat{\Omega}^m(X) = \underbrace{\hat{\Omega}(\hat{\Omega}(\hat{\Omega}\cdots}_{m \text{ times}}(X)\cdots)).$$

Pertaining to the map $\hat{\Omega}$, we have the following two very important facts that follow directly from Proposition 8.

Corollary 10. *Suppose $X \in \mathbb{P}^n$ and $Y \in \hat{\mathbb{D}}$.*

(i) *If $X \leqslant Y$, then $X \in \hat{\mathbb{D}}$ and $\hat{\Omega}(X) \leqslant \hat{\Omega}(Y)$;*

(ii) *If* $\hat{\Omega}(Y) \leqslant Y$, *then*, *for all* $m \geqslant 1$, *the following is true*:

$$\hat{\Omega}^{m+1}(Y) \leqslant \hat{\Omega}^{m}(Y) \leqslant Y.$$

Part (i) of the claim follows routinely by an iterative application of Proposition 8; Part (ii) is easily shown by applying Part (i). We accordingly omit the proof.

A very useful corollary of Propositions 8 and 9 follows; recall that k and q are fixed in this section.

Corollary 11. *Suppose that* $X \in \mathscr{X}$ *and satisfies* (2). *Then*, *for all* $m \geqslant 1$, $X_{k+mq} \in \hat{\mathbb{D}}$ *and* $\hat{\Omega}(X_{k+mq}) < X_{k+(m-1)q}$.

The proof is immediate and so is not included.

4.3. Main results

Now we can state the main result of this section.

Theorem 12. *Suppose block-diagonal operators* A, B, C *and* D *are* (k,q)*-eventually periodic. Then* $I - ZA$ *is non-singular and* $\|C(I-ZA)^{-1}ZB + D\| < 1$ *holds if and only if there exists a* (k,q)*-eventually periodic operator* $X_{\text{eper}} \in \mathscr{X}$ *such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^{*} \begin{bmatrix} X_{\text{eper}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X_{\text{eper}} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{13}$$

Proof. The "if" direction follows immediately from Lemma 3.

To prove "only if", we begin by recalling from the proof of Lemma 7 that, without loss of generality, we may assume the length of the finite horizon k to be equal to 1. Then the state space operator A will have the following representation:

$$A = \text{diag}(A_0, \hat{A}_{\text{per}}, \hat{A}_{\text{per}}, \ldots),$$

where $\hat{A}_{\text{per}}$ is defined as in (6). Similar representations apply for B, C, and D.

Now, by assumption, inequality (2) has a solution in $\mathscr{X}$. Then, by invoking Lemma 7, there exists an eventually q-periodic operator X satisfying the aforementioned inequality such that, for some non-negative integer N,

$$X = \text{diag}(X_0, X_1, \ldots, X_{Nq}, \hat{X}, \hat{X}, \ldots),$$

where $\hat{X} = \text{diag}(X_{Nq+1}, \ldots, X_{(N+1)q})$. Invoking Proposition 9 and Corollary 11, we deduce that (5) holds only if the following sequence of inequalities holds:

$$\begin{aligned} &\Omega_0(X_1) < X_0, \\ &\hat{\Omega}(X_{q+1}) < X_1, \\ &\qquad \vdots \\ &\hat{\Omega}(X_{Nq+1}) < X_{(N-1)q+1}, \\ &\hat{\Omega}(X_{Nq+1}) < X_{Nq+1}. \end{aligned} \tag{14}$$

Starting with the second last inequality, we can successively apply Part (i) of Corollary 11 to obtain the inequality

$$\hat{\Omega}^N(X_{Nq+1}) < X_1.$$

Invoking Proposition 8, the preceding inequality, along with the inequality $\Omega_0(X_1) < X_0$ from (14), guarantees the validity of the following:

$$\Omega_0(\hat{\Omega}^N(X_{Nq+1})) < X_0.$$

Applying the Schur complement formula to the above inequality, we get

$$F_0^* \begin{bmatrix} \hat{\Omega}^N(X_{Nq+1}) & 0 \\ 0 & I \end{bmatrix} F_0 - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < 0,$$

where F_0 is defined as in (11). Note that $\hat{\Omega}^N(X_{Nq+1})$ is positive semi-definite. But, in order for $\hat{\Omega}^N(X_{Nq+1})$ to be part of the solution of inequality (2), it has to be positive definite, and so, we perturb it to achieve this. Choose $0 < \varepsilon < 1$ such that the following strict inequality holds:

$$F_0^* \begin{bmatrix} (1-\varepsilon)\hat{\Omega}^N(X_{Nq+1}) + \varepsilon X_{Nq+1} & 0 \\ 0 & I \end{bmatrix} F_0 - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < 0. \tag{15}$$

Note that $(1-\varepsilon)\hat{\Omega}^N(X_{Nq+1}) + \varepsilon X_{Nq+1} > 0$.

Now invoking part (ii) of Corollary 10, we have

$$\hat{\Omega}^{N+1}(X_{Nq+1}) \leqslant \hat{\Omega}^N(X_{Nq+1}).$$

Defining $\Gamma_i(\hat{\Omega}^N(X_{Nq+1})) = \Omega_i(\Omega_{i+1}(\cdots \Omega_q(\hat{\Omega}^N(X_{Nq+1}))\cdots))$ for $i=2,\dots,q$, we can equivalently write the above inequality as

$$\Omega_1(\Gamma_2(\hat{\Omega}^N(X_{Nq+1}))) \leqslant \hat{\Omega}^N(X_{Nq+1}).$$

Applying the Schur complement to this inequality, we get

$$F_1^* \begin{bmatrix} \Gamma_2(\hat{\Omega}^N(X_{Nq+1})) & 0 \\ 0 & I \end{bmatrix} F_1 - \begin{bmatrix} \hat{\Omega}^N(X_{Nq+1}) & 0 \\ 0 & I \end{bmatrix} \leqslant 0. \tag{16}$$

Also, applying the Schur complement to the inequality $\Gamma_i(\hat{\Omega}^N(X_{Nq+1})) \leqslant \Gamma_i(\hat{\Omega}^N(X_{Nq+1}))$ for $i = 2,\dots,q$, we get

$$F_j^* \begin{bmatrix} \Gamma_{j+1}(\hat{\Omega}^N(X_{Nq+1})) & 0 \\ 0 & I \end{bmatrix} F_j - \begin{bmatrix} \Gamma_j(\hat{\Omega}^N(X_{Nq+1})) & 0 \\ 0 & I \end{bmatrix} \leqslant 0 \tag{17}$$

for $j = 2,\dots,q-1$, and

$$F_q^* \begin{bmatrix} \hat{\Omega}^N(X_{Nq+1}) & 0 \\ 0 & I \end{bmatrix} F_q - \begin{bmatrix} \Gamma_q(\hat{\Omega}^N(X_{Nq+1})) & 0 \\ 0 & I \end{bmatrix} \leqslant 0. \tag{18}$$

Now we can write matrix inequalities (16)–(18) more compactly as

$$\begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix}^* \begin{bmatrix} \hat{Z}^* T \hat{Z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix} - \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \leqslant 0, \tag{19}$$

where

$$T = \operatorname{diag}(\hat{\Omega}^N(X_{Nq+1}), \Gamma_2(\hat{\Omega}^N(X_{Nq+1})), \dots, \Gamma_q(\hat{\Omega}^N(X_{Nq+1}))).$$

At this point, we note that since the eventually q-periodic operator X solves (2), the following inequality holds:

$$\begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix}^* \begin{bmatrix} \hat{Z}^*\hat{X}\hat{Z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix} - \begin{bmatrix} \hat{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{20}$$

Taking the convex combination $\{\varepsilon \times (20) + (1-\varepsilon) \times (19)\}$, we get

$$\begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix}^* \begin{bmatrix} \hat{Z}^*\hat{X}_{\text{per}}\hat{Z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{\text{per}} & \hat{B}_{\text{per}} \\ \hat{C}_{\text{per}} & \hat{D}_{\text{per}} \end{bmatrix} - \begin{bmatrix} \hat{X}_{\text{per}} & 0 \\ 0 & I \end{bmatrix} < 0, \tag{21}$$

where $\hat{X}_{\text{per}} = (1-\varepsilon)T + \varepsilon\hat{X} > 0$. Now it is apparent from inequalities (15) and (21) that the (k,q)-eventually periodic operator ($k=1$ in this case) $X_{\text{eper}} = \text{diag}(X_0, \hat{X}_{\text{per}}, \hat{X}_{\text{per}}, \ldots)$ solves inequality (13). Thus, we have shown that, given (k,q)-eventually periodic block-diagonal state space operators, we can always construct from any solution of inequality (2) a (k,q)-eventually periodic block-diagonal solution. □

Before stating the next result, we require some additional notation. Suppose Q is a (k,q)-eventually periodic block-diagonal operator, then we define $\tilde{Q}$ to be the *finite-horizon-first-period truncation* of Q, namely

$$\tilde{Q} := \text{diag}(Q_0, \ldots, Q_{k-1}, Q_k, \ldots, Q_{k+q-1}),$$

which is a matrix. Also, we define the shift matrices Z_1 and Z_2 for $i,j = 1,\ldots,k+q$ by

$$Z_1 = [a_{ij}], \quad \text{where} \quad a_{ij} = \begin{cases} I & \text{if } i = 2,\ldots,k+q, \; j = i-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$Z_2 = [b_{ij}], \quad \text{where} \quad b_{ij} = \begin{cases} I & \text{if } i = k+1, \; j = k+q, \\ 0 & \text{otherwise}. \end{cases}$$

And so for a block-diagonal matrix $\tilde{Q}$, we have

$$Z_1^*\tilde{Q}Z_1 = \text{diag}(Q_1, \ldots, Q_{k+q-1}, 0) \quad \text{and} \quad Z_2^*\tilde{Q}Z_2 = \text{diag}(0, \ldots, 0, Q_k).$$

Last, define the truncation of the set $\mathscr{X}$, defined in (3), by

$$\tilde{\mathscr{X}} := \{\tilde{X}: X \in \mathscr{X}\}.$$

Using these new definitions, we have the following corollary of Theorem 12 and Lemma 3.

Corollary 13. *Suppose block-diagonal operators A, B, C and D are (k,q)-eventually periodic. The following conditions are equivalent:*

(i) $\|C(I-ZA)^{-1}ZB + D\| < 1$ *and $I - ZA$ is invertible*;
(ii) *There exists a matrix $\tilde{X} \in \tilde{\mathscr{X}}$ such that*

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^* \begin{bmatrix} Z_1^*\tilde{X}Z_1 + Z_2^*\tilde{X}Z_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} - \begin{bmatrix} Z_1^*\tilde{X}Z_1 + Z_2^*\tilde{X}Z_2 & 0 \\ 0 & I \end{bmatrix} < 0. \tag{22}$$

Thus, this corollary gives a finite dimensional convex condition for determining the ℓ_2 induced norm of an eventually periodic system with block-diagonal state space operators A, B, C, and D. This condition can be checked using various convex programming techniques; see for example [3] for a synopsis of such methods.

5. Synthesis application: minimizing the ℓ_2-induced norm

Let G be a linear time-varying discrete-time system defined by the following state space equation:

$$\begin{bmatrix} x_{t+1} \\ z_t \\ y_t \end{bmatrix} = \begin{bmatrix} A_t & B_{1t} & B_{2t} \\ C_{1t} & D_{11t} & D_{12t} \\ C_{2t} & D_{21t} & 0 \end{bmatrix} \begin{bmatrix} x_t \\ w_t \\ u_t \end{bmatrix}, \quad x_0 = 0, \tag{23}$$

for $w \in \ell_2$. The signals x_t, z_t, w_t, y_t, and u_t are real and have time-varying dimensions which we denote by n_t, n_{zt}, n_{wt}, n_{yt}, and n_{ut}, respectively. For notational simplicity, in the following we are going to suppress the time-dependence of the above dimensions. We make the assumption that all the state space matrices are uniformly bounded functions of time, and further assume the direct feedthrough term $D_{22} = 0$. Also, we assume that the block-diagonal operators, defined by the sequences of the above state space matrices, are (k, q)-eventually periodic.

We suppose this system is being controlled by a controller K whose state space equation is

$$\begin{bmatrix} x_{t+1}^K \\ u_t \end{bmatrix} = \begin{bmatrix} A_t^K & B_t^K \\ C_t^K & D_t^K \end{bmatrix} \begin{bmatrix} x_t^K \\ y_t \end{bmatrix}, \quad x_0^K = 0.$$

The controller state vector $x_t^K \in \mathbb{R}^r$ where the time dependence of r is suppressed. Again, we assume that the block-diagonal operators, defined by the matrix sequences A_t^K, B_t^K, C_t^K, and D_t^K, are (k, q)-eventually periodic. The connection of G and K is shown in Fig. 1. Since $D_{22} = 0$, this interconnection is always well-posed.

We write the realization of the closed loop system as

$$\begin{aligned} x_{t+1}^L &= A_t^L x_t + B_t^L w_t, \\ z_t &= C_t^L x_t + D_t^L w_t, \end{aligned} \tag{24}$$

where x_t^L contains the combined states of G and K, and A_t^L, B_t^L, C_t^L and D_t^L are appropriately defined. Here $A_t^L \in \mathbb{R}^{(n+r)\times(n+r)}$, where n is the number of states of G and r is the number of states of K. Note that the block-diagonal operators A^L, B^L, C^L and D^L are (k, q)-eventually periodic.

The following definition expresses our synthesis goal.

Definition 14. A controller K is an *admissible synthesis* for G in Fig. 1 if $I - ZA^L$ is invertible and the closed-loop performance inequality $\|w \mapsto z\|_{\ell_2 \to \ell_2} < 1$ is achieved.

The development of the solution of the above synthesis problem is very similar to the ones presented in [7,13] for the time-invariant case, and the one given in [5] for the time-varying case. Hence, it would be

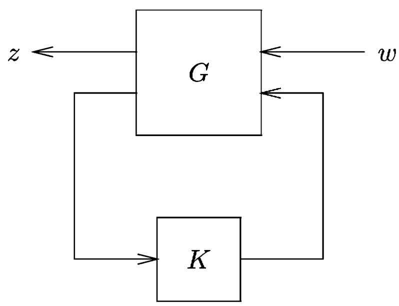

Fig. 1. Closed-loop system.

very repetitive to include it here, and so, we are only going to present the main result. But before that, we have to introduce one more definition. Given a block-diagonal operator $X \in \mathscr{X}$, we define the block-diagonal operators R and S via

$$X = \left[\!\left[\begin{matrix} S & N \\ N^* & ? \end{matrix} \right]\!\right], \quad X^{-1} = \left[\!\left[\begin{matrix} R & L \\ L^* & ? \end{matrix} \right]\!\right], \tag{25}$$

where $R_t, S_t \in \mathbb{R}^{n\times n}$ and $L_t, N_t \in \mathbb{R}^{n\times r}$.

The following theorem has conditions that only depend on the plant data and are independent of r, the controller state dimension; more importantly these conditions are convex and finite dimensional.

Theorem 15. *Suppose that plant G is (k,q)-eventually periodic. Then there exists an admissible (k,q)-eventually periodic synthesis K for G with state dimension $r \geqslant n$ if and only if there exist block-diagonal matrices $\tilde{R} > 0$ and $\tilde{S} > 0$ satisfying*

$$\text{(i)} \quad \begin{bmatrix} \tilde{N}_R & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \tilde{A}\tilde{R}\tilde{A}^* - Z_1^*\tilde{R}Z_1 - Z_2^*\tilde{R}Z_2 & \tilde{A}\tilde{R}\tilde{C}_1^* & \tilde{B}_1 \\ \tilde{C}_1\tilde{R}\tilde{A}^* & \tilde{C}_1\tilde{R}\tilde{C}_1^* - I & \tilde{D}_{11} \\ \tilde{B}_1^* & \tilde{D}_{11}^* & -I \end{bmatrix} \begin{bmatrix} \tilde{N}_R & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\text{(ii)} \quad \begin{bmatrix} \tilde{N}_S & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \tilde{A}^*(Z_1^*\tilde{S}Z_1 + Z_2^*\tilde{S}Z_2)\tilde{A} - \tilde{S} & \tilde{A}^*(Z_1^*\tilde{S}Z_1 + Z_2^*\tilde{S}Z_2)\tilde{B}_1 & \tilde{C}_1^* \\ \tilde{B}_1^*(Z_1^*\tilde{S}Z_1 + Z_2^*\tilde{S}Z_2)\tilde{A} & \tilde{B}_1^*(Z_1^*\tilde{S}Z_1 + Z_2^*\tilde{S}Z_2)\tilde{B}_1 - I & \tilde{D}_{11}^* \\ \tilde{C}_1 & \tilde{D}_{11} & -I \end{bmatrix} \begin{bmatrix} \tilde{N}_S & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\text{(iii)} \quad \begin{bmatrix} \tilde{R} & I \\ I & \tilde{S} \end{bmatrix} \geqslant 0,$$

where the operators $\tilde{N}_R$, $\tilde{N}_S$ satisfy

$$\operatorname{Im} \tilde{N}_R = \operatorname{Ker}[\tilde{B}_2^* \quad \tilde{D}_{12}^*], \qquad \tilde{N}_R^*\tilde{N}_R = I,$$

$$\operatorname{Im} \tilde{N}_S = \operatorname{Ker}[\tilde{C}_2 \quad \tilde{D}_{21}], \qquad \tilde{N}_S^*\tilde{N}_S = I.$$

The theorem states that the validity of the above convex synthesis conditions is equivalent to the existence of an admissible (k,q)-eventually periodic synthesis K for plant G. Solutions $\tilde{R}$ and $\tilde{S}$ can be used to construct a (k,q)-eventually periodic controller K. The way to construct this controller can be found in [5,7,13].

Remark 16. If the synthesis conditions in Theorem 15 are invalid, we can only say that there exists no admissible (k,q)-*eventually periodic synthesis*; but this does not necessarily imply the non-existence of a different admissible synthesis. In fact, it is not difficult to construct counter examples of (k,q)-eventually periodic plants that admit no (k,q)-*but rather* (N,q)-eventually periodic syntheses, where $N > k$.

Remark 17. A (k,q)-eventually periodic plant G is also (N,q)-eventually periodic for all integers $N \geqslant k$. Thus, if no admissible (k,q)-eventually periodic synthesis for G exists, we may still utilize the synthesis conditions of Theorem 15 as part of an algorithm to find an admissible (M,q)-eventually periodic controller, where M is the minimum integer greater or equal to k such that the aforementioned synthesis conditions hold for (M,q)-eventually periodic plant G.

In the following subsection, we show how these synthesis results can be used in situations where there are nonzero initial conditions.

5.1. Control of systems with uncertain initial conditions

In this subsection, we illustrate how the results of the paper can be applied to linear systems with uncertain initial conditions. The approach can be applied to general eventually periodic systems; however, to keep things simple, in this subsection we will focus on the case where the nominal system is linear time-invariant. We note that related results to this special case can be found in [4,12].

Given an LTI system with nonzero initial condition $\bar{x}_0$, we seek a stabilizing LTI controller that renders the closed-loop map $(\bar{x}_0, w) \mapsto \bar{z}$ contractive, where the input channel $\bar{w}$ and the output channel $\bar{z}$ represent the exogenous disturbances and exogenous errors, respectively; hence, we require a stabilizing LTI controller such that

$$\sup_{\|\bar{x}_0\|_{\mathbb{R}^n}+\|\bar{w}\|_{\ell_2}\neq 0} \frac{\|\bar{z}\|_{\ell_2}}{\sqrt{\|\bar{x}_0\|^2_{\mathbb{R}^n}+\|\bar{w}\|^2_{\ell_2}}} < 1. \tag{26}$$

For that purpose, we make use of the fact that, given any LTI plant L having a nonzero initial condition, we can always construct from it an *isomorphic* k-eventually time-invariant system G with the finite horizon length $k = 1$, and vice versa. Now Theorem 15 already gives us finite dimensional convex conditions that are necessary and sufficient for the existence of an admissible k-eventually time-invariant controller for system G. Then, because of the isomorphism between systems L and G, the validity of the above conditions is also equivalent to the existence of an LTI synthesis for system L.

Now let system G be defined as in (23), and assume that the block-diagonal operators constructed from the state space matrix sequences are k-eventually time-invariant with the finite horizon length $k = 1$. Furthermore, assume that the input signal w_t has dimension $n+n_w$ at time $t=0$ as opposed to just n_w, its constant dimension at all other times. Set $w_0 = [\bar{x}_0^{\mathrm{T}} \quad ?]^{\mathrm{T}}$, where $w_0 \in \mathbb{R}^{n+n_w}$ and $\bar{x}_0 \in \mathbb{R}^n$. Also, suppose that all the state space matrices at time $t = 0$ are zeroes except for B_{10} which is taken equal to $[I_n \quad 0_{n\times n_w}]$. Now since we have chosen our system state space matrices at $t = 0$ such that $y_0 = 0$ and $z_0 = 0$, the above system is isomorphic to the following LTI system L:

$$\begin{bmatrix} \bar{x}_{t+1} \\ \bar{z}_t \\ \bar{y}_t \end{bmatrix} = \begin{bmatrix} A_1 & B_{11} & B_{21} \\ C_{11} & D_{111} & D_{121} \\ C_{21} & D_{211} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{w}_t \\ \bar{u}_t \end{bmatrix}, \tag{27}$$

where $\bar{x}_t = x_{t+1}$, $\bar{u}_t = u_{t+1}$, $\bar{y}_t = y_{t+1}$, $\bar{w}_t = w_{t+1}$, and $\bar{z}_t = z_{t+1}$ for all non-negative integers t. Hence, G is isomorphic to LTI system L that has an uncertain initial state $\bar{x}_0 = x_1$, and the isomorphism is clear from (27). Conversely, starting with an LTI plant having a non-zero initial condition such as L, we can construct a k-eventually time-invariant system G with $k = 1$ following the above steps reversely. Then we apply Theorem 15, and if the synthesis conditions are valid, we can build a k-eventually time-invariant controller K, whose time-invariant portion (with a zero initial condition) constitutes an admissible LTI synthesis for LTI system L.

We note here that the norm used on the initial uncertain state $\bar{x}_0$ and the input $\bar{w}$, as defined in (26), couples the two objects. In many situations, a more natural scenario is to have the norm constraints placed on $\bar{x}_0$ and $\bar{w}$ independent; for instance, we would like to pose a problem where $\bar{x}_0$ is known to reside in a norm ball of radius say ε, and $\bar{w}$ is constrained to satisfy an independent condition of the form $\|\bar{w}\|_{\ell_2} \leqslant \alpha$. It is possible to combine the result in Theorem 15 with the results in [14] to get systematically obtained controllers for this purpose.

6. Conclusions

In this paper, we have introduced the notion of a (k,q)-eventually periodic linear system. We have derived a version of the KYP lemma which characterizes open-loop stability and performance of such systems. For closed-loop systems, we have provided necessary and sufficient LMI conditions for the existence and construction of (k,q)-eventually periodic controllers given (k,q)-eventually periodic plants. We have also asserted that, in this scenario, the optimal controller may not in general be (k,q)-eventually periodic; however, the optimal performance can be approached arbitrarily by increasing the finite horizon variable k. Finally, we have explicitly illustrated how this synthesis result can also be used to find controllers in the presence of uncertain initial states.

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