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Duality and eventually periodic systems

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SUMMARY

This paper employs semidefinite programming duality theory to develop new alternative linear matrix inequality (LMI) tools for *eventually periodic* systems. These tools are then utilized to rederive an important version of the Kalman–Yakubovich–Popov (KYP) Lemma for such systems, and further give new synthesis results. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: time-varying systems; orbits; LMIs; duality; ℓ_2-induced control; eventually periodic systems

1. INTRODUCTION

In this paper, we continue our work started in Reference [1] on the control of *eventually periodic* systems. Such systems are aperiodic for an initial amount of time, and then become periodic afterwards. Eventually periodic dynamics arise in various scenarios. One of these scenarios is when linearizing a system along a trajectory composed of an aperiodic manoeuver and a subsequent periodic orbit. Another is when considering plants with uncertain initial states. It is worth noting that both finite horizon and periodic systems are subclasses of eventually periodic systems.

Primarily, this paper serves as a gateway for the use of semidefinite programming duality results in control problems involving eventually periodic systems. In fact, we will show that all analysis and synthesis convex conditions pertaining to the ℓ_2-induced control of such systems can be provided in terms of finite-dimensional semidefinite programming problems. Then, by appealing to the vast literature on duality, it is possible to develop new alternative tools that would potentially offer new theoretical insight and possibly help provide new results in this area of research. Specifically, in this paper, we will invoke one of the theorems of alternatives of Reference [2], which is itself a special case of the Hahn–Banach separation theorem, to develop new alternative linear matrix inequality (LMI) tools that would be later used to rederive an

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important version of the Kalman–Yakubovich–Popov (KYP) Lemma for eventually periodic systems, originally proved in Reference [1], and further give new synthesis results.

The main contributions of this paper are:

- establishing that the existence of a synthesis for an eventually periodic plant is always equivalent to the existence of an eventually periodic synthesis, having the same periodicity as the plant but probably exhibiting longer transient time variation;
- using alternative LMI tools, which stem from semidefinite programming duality theory to (1) give a simpler derivation of an important version of the KYP lemma for eventually periodic systems; and (2) closely study each of the synthesis conditions, highlighting consequently certain cases where a synthesis if existent can always be chosen to be of the same eventually periodic class as the plant.

Note that most of the synthesis results of this paper are also given in the conference paper [3], but the alternative proofs herein are overall far simpler and more concise.

The literature on semidefinite programming is vast; some of the seminal papers on this subject are References [4–6], and we refer the reader to Reference [2] for further references. In addition to its applications in control, semidefinite programming has lots of applications in combinatorial optimization; see for instance Reference [7]. Also, some important references on convex optimization problems and associated duality theory are References [8, 9]. The general machinery used to obtain the results of this paper is motivated by the work in References [2, 10–12], combined with the time-varying system machinery developed in Reference [13]. There is a rich literature in the area of time-varying systems, and a comprehensive list of general references can be found in Reference [14]. Finally, while Reference [2] constitutes the main inspiration for this work, it is worth noting that, as stated in Reference [2], there are other papers that utilize notions from convex optimization duality to give new proofs of existing results (see, for example, Reference [15]) or derive new ones [16].

2. PRELIMINARIES

We now introduce our notation and gather some elementary facts. The set of real numbers and that of real $n \times m$ matrices are denoted by $\mathbb{R}$ and $\mathbb{R}^{n\times m}$, respectively. Given a square matrix $X \in \mathbb{R}^{n\times n}$, the dimension of X is $\dim(X) = n$. The image space and kernel of a linear mapping A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. If S_i is a sequence of operators, then $\operatorname{diag}(S_i)$ denotes their block-diagonal augmentation.

Given two Hilbert spaces E and F, we denote the space of bounded linear operators mapping E to F by $\mathscr{L}(E, F)$, and shorten this to $\mathscr{L}(E)$ when E equals F. If X is in $\mathscr{L}(E, F)$, we denote the E to F induced norm of X by $\|X\|_{E\to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. When an operator $X \in \mathscr{L}(E)$ is self-adjoint, we use $X \prec 0$ to mean it is negative definite; that is there exists a number $\alpha > 0$ such that, for all non-zero $x \in E$, the inequality $\langle x, Xx\rangle < -\alpha\|x\|^2$ holds, where $\langle\cdot,\cdot\rangle$ denotes the inner product and $\|\cdot\|$ denotes the corresponding norm on E.

The main Hilbert space of interest in this paper is formed from an infinite sequence of Euclidean spaces $J = (\mathbb{R}^{n_0}, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \ldots)$, and is denoted by $\ell_2(J)$. It consists of elements $x = (x_0, x_1, x_2, \ldots)$, with each $x_k \in \mathbb{R}^{n_k}$, such that $\|x\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2 < \infty$. The inner product of x, y

in $\ell_2(J)$ is hence defined as the sum $\langle x, y\rangle_{\ell_2} = \sum_{k=0}^{\infty} \langle x_k, y_k\rangle_{\mathbb{R}^{n_k}}$. If the sequence of spaces J is clear from the context, then the notation $\ell_2(J)$ is abbreviated to ℓ_2.

One of the most important operators used in the paper is the unilateral shift operator Z defined as follows:

$$Z : \ell_2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \ldots) \to \ell_2(\mathbb{R}^{n_0}, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \ldots)$$

$$(a_1, a_2, \ldots) \overset{Z}{\mapsto} (0, a_1, a_2, \ldots)$$

Following the notation and approach in Reference [13], we make the following definitions.

Definition 1

A bounded linear operator Q mapping $\ell_2(\mathbb{R}^{m_0}, \mathbb{R}^{m_1}, \ldots)$ to $\ell_2(\mathbb{R}^{n_0}, \mathbb{R}^{n_1}, \ldots)$ is *block-diagonal* if there exists a sequence of matrices Q_k in $\mathbb{R}^{n_k \times m_k}$ such that, for all w, z, if $z = Qw$, then $z_k = Q_k w_k$. Then Q has the representation $\mathrm{diag}(Q_0, Q_1, Q_2, \ldots)$.

Definition 2

An operator P on ℓ_2 is (h, q)-eventually periodic if, for some integers $h \geqslant 0$ and $q \geqslant 1$, we have

$$Z^q((Z^*)^h P Z^h) = ((Z^*)^h P Z^h) Z^q \tag{1}$$

In the case where the finite horizon length h is not known or when only the period length q is relevant, we simply refer to P as *eventually q-periodic*.

Note that when $h = 0$, equality (1) reduces to the following: $Z^q P = P Z^q$. Hence, in such a case, P simply commutes with the q-shift, and we accordingly refer to P as a q-periodic operator. Throughout the sequel we set $h \geqslant 0$ and $q \geqslant 1$ to be some fixed integers.

We denote the *first period truncation* of a q-periodic block-diagonal operator Q by $\hat{Q}$, and define such a matrix as $\hat{Q} := \mathrm{diag}(Q_0, \ldots, Q_{q-1})$. Also, we define the cyclic shift matrix $\hat{Z}$ for $q \geqslant 2$ by

$$\hat{Z} = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}$$

so that $\hat{Z}^* \hat{Q} \hat{Z} = \mathrm{diag}(Q_1, \ldots, Q_{q-1}, Q_0)$. For $q = 1$, we set $\hat{Z} = I$.

Now, suppose that Q is an (N, q)-eventually periodic block-diagonal operator, then we define $\tilde{Q}$ to be the (N, q)-truncation of Q, namely $\tilde{Q} := \mathrm{diag}(Q_0, Q_1, \ldots, Q_{N+q-1})$, which is a matrix. Also, we define the shift matrices Z_1 and Z_2 for $i, j = 1, \ldots, N+q$ by

$$Z_1 = [a_{ij}] \quad \text{where } a_{ij} = \begin{cases} I & \text{if } i = 2, \ldots, N+q, \; j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

$$Z_2 = [b_{ij}] \quad \text{where } b_{ij} = \begin{cases} I & \text{if } i = N+1, \; j = N+q \\ 0 & \text{otherwise} \end{cases}$$

For $N=0, q=1$, we set $Z_1=0$. And so, excluding the preceding case, given a block-diagonal matrix $\tilde{Q}$, we have $Z_1^*\tilde{Q}Z_1=\text{diag}(Q_1,\ldots,Q_{N+q-1},0)$ and $Z_2^*\tilde{Q}Z_2=\text{diag}(0,\ldots,0,Q_N)$.

Having established these definitions, we are now ready to consider the main subject of this paper.

3. EVENTUALLY PERIODIC PLANTS AND LMI TOOLS

This section is divided into two subsections. The first formulates the ℓ_2-induced control problem for eventually periodic systems, and the second reviews some useful analysis and synthesis results from References [1, 13] pertaining to this problem.

3.1. Problem formulation

Let G be a linear time-varying (LTV) discrete-time system defined by the state space equation

$$\begin{bmatrix} x_{k+1} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A_k & B_{1k} & B_{2k} \\ C_{1k} & D_{11k} & D_{12k} \\ C_{2k} & D_{21k} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix} \quad x_0 = 0 \tag{2}$$

for $w \in \ell_2$, where the block-diagonal operators, defined by the sequences of the above state space matrices, are (h,q)-eventually periodic. Because of such eventually periodic dynamics, we refer to G as an (h,q)-eventually periodic plant. The input channels into plant G are the exogenous disturbances w and the applied control u, and the corresponding output channels are the exogenous errors z and the measurements y, respectively. The signals x_k, z_k, w_k, y_k, and u_k are real and have time-varying dimensions which we denote by $n_k, n_{zk}, n_{wk}, n_{yk}$, and n_{uk}, respectively.

We suppose this system is being controlled by an LTV controller K whose state space equation is

$$\begin{bmatrix} x_{k+1}^K \\ u_k \end{bmatrix} = \begin{bmatrix} A_k^K & B_k^K \\ C_k^K & D_k^K \end{bmatrix} \begin{bmatrix} x_k^K \\ y_k \end{bmatrix} \quad x_0^K = 0$$

where $x_k^K \in \mathbb{R}^{r_k}$. The connection of G and K is shown in Figure 1. Since $D_{22}=0$, this interconnection is always well posed.

We write the realization of the closed-loop system as

$$\begin{aligned} x_{k+1}^L &= A_k^L x_k^L + B_k^L w_k \\ z_k &= C_k^L x_k^L + D_k^L w_k \end{aligned} \tag{3}$$

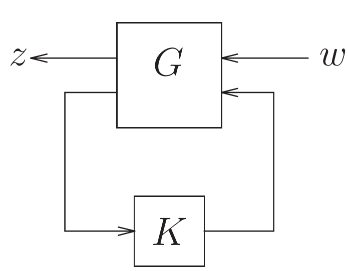

Figure 1. Closed-loop system.

where $x_k^L \in \mathbb{R}^{n_k+r_k}$ contains the combined states of G and K at time k, and A_k^L, B_k^L, C_k^L and D_k^L are appropriately defined. This closed-loop system may be written more compactly in operator form as

$$\begin{aligned} x^L &= ZA^L x^L + ZB^L w \\ z &= C^L x^L + D^L w \end{aligned} \tag{4}$$

where Z is the shift operator on ℓ_2. Assuming the relevant inverse exists, we can write the map from w to z as

$$w \mapsto z = C^L(I - ZA^L)^{-1} ZB^L + D^L$$

We will say this closed-loop state space system is stable when $I - ZA^L$ has a bounded inverse; this is equivalent to exponential stability as shown in Reference [13]. In the case of an eventually periodic controller, the closed-loop system would be eventually periodic as well, and so its stability boils down to the stability of its periodic part.

The following definition expresses our synthesis goal.

Definition 3
A controller K is an *admissible synthesis* for plant G in Figure 1 if $I - ZA^L$ has a bounded inverse and the closed-loop performance inequality $\|w \mapsto z\|_{\ell_2 - \ell_2} < 1$ is achieved.

3.2. Analysis and synthesis results

We now briefly review some analysis and synthesis results; see References [1, 13] for an in-depth presentation. To start, we define the set $\mathscr{X}^L$ to consist of all the positive definite block-diagonal operators $X \in \mathscr{L}(\ell_2)$ of the form $X = \operatorname{diag}(X_0, X_1, \ldots)$, where $X_i \in \mathbb{R}^{(n_i+r_i)\times(n_i+r_i)}$. Similarly, we define the set $\mathscr{X}$, except that here the matrix blocks $X_i \in \mathbb{R}^{n_i \times n_i}$. Following is the KYP lemma for LTV models as given in Reference [13].

Lemma 4
Suppose operators A^L, B^L, C^L, and D^L are block-diagonal. The following conditions are equivalent:

(i) $\|C^L(I - ZA^L)^{-1} ZB^L + D^L\| < 1$ and $I - ZA^L$ has a bounded inverse;
(ii) There exists $X \in \mathscr{X}^L$ such that

$$\begin{bmatrix} ZA^L & ZB^L \\ C^L & D^L \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA^L & ZB^L \\ C^L & D^L \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{5}$$

In the event that the controller state space operators are (N, q)-eventually periodic for some integer $N \geqslant h$, then the closed-loop state space operators A^L, B^L, C^L, and D^L are also (N, q)-eventually periodic, and condition (ii) in the above lemma can be further strengthened by imposing additional structure on the operator X. In fact, Reference [1] shows that, in such a case, a solution in $\mathscr{X}^L$ exists to inequality (5) if and only if an (N, q)-eventually periodic solution in $\mathscr{X}^L$ exists. Later in this paper, we will present another proof to the previous statement using one of the theorems of alternatives given in Reference [2]. Following is the main synthesis result from Reference [13] for LTV systems.

Theorem 5
There exists an admissible LTV synthesis K for (h,q)-eventually periodic plant G, with state dimension $r_k \leqslant n_k$ for all k, if and only if there exist operators $R, S \in \mathscr{X}$ satisfying

$$F^* R F - V_1^* Z^* R Z V_1 + H \prec 0 \tag{6}$$

$$J^* Z^* S Z J - U_1^* S U_1 + W \prec 0 \tag{7}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \succcurlyeq 0 \tag{8}$$

where

$$\begin{aligned} &\operatorname{Im}[V_1^* \; V_2^*]^* = \operatorname{Ker}[B_2^* \; D_{12}^*], \quad [V_1^* \; V_2^*]\,[V_1^* \; V_2^*]^* = I \\ &\operatorname{Im}[U_1^* \; U_2^*]^* = \operatorname{Ker}[C_2 \; D_{21}], \quad [U_1^* \; U_2^*]\,[U_1^* \; U_2^*]^* = I \end{aligned} \tag{9}$$

and

$$\begin{aligned} &F = A^* V_1 + C_1^* V_2, \quad M = B_1^* V_1 + D_{11}^* V_2, \quad H = M^* M - V_2^* V_2 \\ &J = A U_1 + B_1 U_2, \quad L = C_1 U_1 + D_{11} U_2, \quad W = L^* L - U_2^* U_2 \end{aligned} \tag{10}$$

Solutions R and S can then be used to construct a controller K, as shown in References [10, 12, 13]. Note that all of the system operators in (6) and (7) are block-diagonal and (h,q)-eventually periodic.

The synthesis conditions in Theorem 5 are convex, yet infinite-dimensional as they pertain to the existence of a general LTV synthesis. However, if we seek an (N,q)-eventually periodic synthesis for some integer $N \geqslant h$, then these inequalities reduce to finite-dimensional conditions as shown in the following synthesis result from Reference [1]; note that an (h,q)-eventually periodic operator is also (N,q)-eventually periodic for all integers $N \geqslant h$.

Theorem 6
Suppose that integer $N \geqslant h$. There exists an admissible (N,q)-eventually periodic synthesis K for (h,q)-eventually periodic plant G, with state dimension $r_k \leqslant n_k$ for all k, if and only if there exist block-diagonal matrices $\tilde{R}, \tilde{S} \in \tilde{\mathscr{X}}$ satisfying

$$\tilde{F}^* \tilde{R} \tilde{F} - \tilde{V}_1^* (Z_1^* \tilde{R} Z_1 + Z_2^* \tilde{R} Z_2) \tilde{V}_1 + \tilde{H} \prec 0 \tag{11}$$

$$\tilde{J}^* (Z_1^* \tilde{S} Z_1 + Z_2^* \tilde{S} Z_2) \tilde{J} - \tilde{U}_1^* \tilde{S} \tilde{U}_1 + \tilde{W} \prec 0 \tag{12}$$

$$\begin{bmatrix} \tilde{R} & I \\ I & \tilde{S} \end{bmatrix} \succcurlyeq 0 \tag{13}$$

where the notation $\tilde{Q}$ denotes the (N,q)-truncation of Q, and the set $\tilde{\mathscr{X}} := \{\tilde{X} : X \in \mathscr{X}\}$.

We remark that if the synthesis conditions in Theorem 6 are invalid, we can only say that there exists no admissible (N, q)-*eventually periodic synthesis*; but this does not necessarily imply the non-existence of a different admissible synthesis, be it a general LTV synthesis, in which case inequalities (6), (7) and (8) would hold, or still an eventually periodic one but with a larger finite horizon length or larger period. In the next section, we will show that, when it comes to eventually periodic plants, we need not worry about infinite-dimensional synthesis conditions and corresponding LTV syntheses generally exhibiting infinite time variation, since, for such plants, an admissible synthesis exists if and only if an admissible eventually periodic synthesis exists, albeit the finite horizon length of such an eventually periodic synthesis is still to be determined in general.

4. FINITE-DIMENSIONAL SEMIDEFINITE PROGRAMMING PROBLEMS

A very appealing feature about the ℓ_2-induced control problem for discrete-time eventually periodic systems is that all the analysis and synthesis convex conditions derived can be provided in terms of finite-dimensional semidefinite programming problems. To elaborate more on this, consider the following linear operator inequality in variable X:

$$\varepsilon_1 P^* Z^* XZP + \varepsilon_2 Q^* XQ + T \prec 0 \tag{14}$$

where $X \succ 0$, P, Q and $T = T^*$ are all block-diagonal operators with compatible matrix block dimensions, and the integers $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. This inequality represents a general form of the analysis condition (5) and synthesis conditions (6) and (7). Let $\mathscr{X}_g$ denote the set of positive definite block-diagonal solutions X, of the same structure as operator QQ^*, satisfying inequality (14). Finding a solution $X \in \mathscr{X}_g$ to (14) is in general an infinite-dimensional semidefinite programming problem. However, if the operators P, Q and T are (h, q)-eventually periodic, then a solution $X \in \mathscr{X}_g$ exists if and only if an (N, q)-eventually periodic solution in $\mathscr{X}_g$ exists for some integer $N \geqslant h$, as we will show in the next proposition. Accordingly, in such a case, we only need to consider a finite number of matrix variables, and hence a finite number of LMIs; yet this number is still to be determined in general.

Proposition 7

Suppose that P, Q, and $T = T^*$ are block-diagonal operators with compatible matrix block dimensions, and integers $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Then the following hold:

(i) If P, Q, and T are q-periodic, then there exists a solution in $\mathscr{X}_g$ to inequality (14) if and only if there exists a q-periodic operator $X \in \mathscr{X}_g$ such that

$$\varepsilon_1 \hat{P}^* \hat{Z}^* \hat{X} \hat{Z} \hat{P} + \varepsilon_2 \hat{Q}^* \hat{X} \hat{Q} + \hat{T} \prec 0$$

(ii) If P, Q, and T are (h, q)-eventually periodic, then there exists a solution in $\mathscr{X}_g$ to inequality (14) if and only if there exists an (N, q)-eventually periodic operator in $\mathscr{X}_g$ for some integer $N \geqslant h$ such that

$$\varepsilon_1 \tilde{P}^* (Z_1^* \tilde{X} Z_1 + Z_2^* \tilde{X} Z_2) \tilde{P} + \varepsilon_2 \tilde{Q}^* \tilde{X} \tilde{Q} + \tilde{T} \prec 0 \tag{15}$$

where the notation $\tilde{Q}$ denotes the (N, q)-truncation of Q.

For space considerations, we will only provide the outline of the proof. The proof of Part (i) employs a similar averaging technique to that used in the proof of Theorem 20 in Reference [13]. As for Part (ii), the proof follows from exactly the same argument as that of the proof of Lemma 7 in Reference [1], which amounts to equivalently rewriting inequality (14) as an infinite sequence of LMIs, each corresponding to a distinct time k, then making use of the continuity and convexity properties of these LMIs, together with Part (i), to construct an eventually q-periodic solution.

As stated earlier, inequality (14) represents a general form of the analysis condition (5) and synthesis conditions (6) and (7), and so, the results of Proposition 7 clearly apply to these conditions. In fact, the said results still apply in the case where we have two inequalities of the form of (14) admitting two solutions $R, S \in \mathcal{X}_g$, which are coupled by condition (8). Specifically, we have the following synthesis result.

Theorem 8

Given an (h, q)-eventually periodic plant G, there exists an admissible synthesis K for G if and only if there exists an admissible (N, q)-eventually periodic synthesis for some integer $N \geqslant h$, which in turn is equivalent to the existence of (N, q)-eventually periodic operators $R, S \in \mathcal{X}$ for some $N \geqslant h$ satisfying the synthesis LMIs (11)–(13).

Proof

The proof of the 'if' direction is immediate. To prove the 'only if' direction, we first note that, since there exists an admissible synthesis for the (h, q)-eventually periodic plant, then definitely there exists an admissible synthesis for the q-periodic portion of this plant. Then invoking Theorem 22 of Reference [13], we deduce that there exists an admissible q-periodic synthesis for this periodic part; the proof uses the same averaging argument as that of Part (i) of Proposition 7, which is given in details in the proof of Theorem 20 in Reference [13]. Then, given q-periodic solutions to the periodic parts of the synthesis conditions, we apply the same argument as that of the proof of Part (ii) of Proposition 7 to each of the synthesis conditions (6) and (7), yet using the same ε and ξ for both cases (see proof of Lemma 7 in Reference [1]), so that, for some integer $N \geqslant h$, the resulting (N, q)-eventually periodic solutions would still satisfy the coupling condition (8). □

The question at this point is whether we can find reasonable values for the finite horizon lengths of such eventually q-periodic syntheses. This is still an open problem in general. We will show later in this paper that, given an (h, q)-eventually periodic plant, a solution in $\mathcal{X}$ exists to the synthesis condition (6) if and only if an (h, q)-eventually periodic solution in $\mathcal{X}$ exists; however, this cannot be said for the synthesis condition (7). Moreover, even if (6) and (7) both admit solutions in the subclass of (N, q)-eventually periodic operators in $\mathcal{X}$ for some $N \geqslant h$, none of these solutions might satisfy the coupling condition (8), and hence we may need to settle for a larger finite horizon length. We will also consider in the sequel a couple of special cases where inequality (7) simplifies significantly and then an admissible synthesis if existent can always be chosen to be (h, q)-eventually periodic.

Remark 9
The results of Proposition 7 can be extended to a more general linear operator inequality than (14), but this is not necessary for this paper. Having said that, it is reasonable to deduce that, given eventually periodic plants, then regardless of the control problem, if the solutions are expressible in terms of linear operator inequalities that can be equivalently written as infinite sequences of LMIs, most probably the arguments of the proofs of Proposition 7 and Theorem 8 can still be utilized to prove similar results for the control problem in question, mainly that the solutions can be equivalently provided in terms of finite-dimensional semidefinite programming problems.

Remark 10
We say a synthesis is γ-admissible if it stabilizes the closed-loop system and further guarantees that $\|w \mapsto z\|_{\ell_2 \to \ell_2} < \gamma$. Clearly, a γ-admissible synthesis for G is an admissible synthesis for $\bar{G}$, where $\bar{G}$ has the same system realization as G except that $\bar{C}_1 = (1/\gamma)C_1, \bar{D}_{11} = (1/\gamma)D_{11}$, and $\bar{D}_{12} = (1/\gamma)D_{12}$. Given a stabilizable and detectable (h, q)-eventually periodic plant G, let γ_N denote the minimum γ, up to a certain tolerance, that is achievable by an (N, q)-eventually periodic synthesis where $N \geqslant h$. Then, it is clear from the preceding and Theorem 6 that the value of γ_N and a corresponding γ_N-admissible (N, q)-eventually periodic synthesis can be obtained by solving the following semidefinite programming optimization problem:

$$\begin{aligned}
&\text{minimize:} \quad \gamma^2 \\
&\text{subject to:} \quad \tilde{F}^*\tilde{R}\tilde{F} - \tilde{V}_1^*(Z_1^*\tilde{R}Z_1 + Z_2^*\tilde{R}Z_2)\tilde{V}_1 + \tilde{M}^*\tilde{M} - \gamma^2\tilde{V}_2^*\tilde{V}_2 \prec 0 \\
&\qquad \begin{bmatrix} \tilde{J}^*(Z_1^*\tilde{S}Z_1 + Z_2^*\tilde{S}Z_2)\tilde{J} - \tilde{U}_1^*\tilde{S}\tilde{U}_1 - \tilde{U}_2^*\tilde{U}_2 & \tilde{L}^* \\ \tilde{L} & -\gamma^2 I \end{bmatrix} \prec 0 \qquad (16) \\
&\qquad \begin{bmatrix} \tilde{R} & I \\ I & \tilde{S} \end{bmatrix} \succcurlyeq 0, \quad \tilde{R}, \tilde{S} \in \tilde{\mathscr{X}}
\end{aligned}$$

where the notation $\tilde{Q}$ denotes the (N, q)-truncation of Q, and the above system matrices are defined in (9) and (10). Solving (16) for an increasing sequence of finite horizon lengths N_i, where $N_0 = h$, results in a non-increasing sequence of optimal values γ_{N_i}. Hence, there is a trade-off in general between N (the 'size' of the synthesis) and γ (the desired performance). Clearly, the sequence γ_{N_i} can serve as a guideline for choosing a synthesis of reasonable size and performance. For example, once the difference between two consecutive elements in this sequence, say γ_{N_j} and $\gamma_{N_{j+1}}$, falls within a certain tolerance, we may then terminate the iteration and use the solutions $\tilde{R}$ and $\tilde{S}$ corresponding to γ_{N_j} to construct a γ_{N_j}-admissible (N_j, q)-eventually periodic synthesis.

5. A STRONG THEOREM OF ALTERNATIVES

As the literature on duality for finite-dimensional semidefinite programming problems is vast, many of the already available duality results can be rephrased to suit our control problem, and, as a result, help provide new theoretical insight and maybe even new results for such a problem. It is worth noting that, in this paper, by invoking only one of the theorems of alternatives of

Reference [2], which is itself a special case of the Hahn–Banach separation theorem, we give a simpler derivation of an important version of the KYP lemma for eventually periodic systems, first given in Reference [1], and further provide new synthesis results, whose proofs are mostly simpler and more concise than the ones given in the conference paper [3]. Hence the appeal of using such duality results. Following is the aforementioned theorem of alternatives from Reference [2].

Theorem 11
Suppose that $\mathscr{V}$ is a finite-dimensional vector space with an inner product $\langle \cdot, \cdot \rangle_{\mathscr{V}}$, and that $\mathscr{S}$ is a space of block-diagonal self-adjoint matrices of the form $X = \mathrm{diag}(X_0, X_1, \ldots, X_s)$, where $X_i = X_i^* \in \mathbb{R}^{m_i \times m_i}$ for some integers $m_i > 0$ and $s \geqslant 0$, and with inner product $\langle X, Y \rangle_{\mathscr{S}} = \mathrm{trace}(XY)$. Then, given a linear mapping $\mathfrak{E} : \mathscr{V} \to \mathscr{S}$, its adjoint mapping $\mathfrak{E}^*$, and a matrix $E_0 \in \mathscr{S}$, exactly one of the following statements is true:

(1) There exists an $x \in \mathscr{V}$ satisfying the LMI $\mathfrak{E}(x) + E_0 \succ 0$;
(2) There exists a *non-zero* $Y \in \mathscr{S}$ such that $Y \succcurlyeq 0$, $\mathfrak{E}^*(Y) = 0$, and $\langle E_0, Y \rangle_{\mathscr{S}} \leqslant 0$.

Remark 12
In the preceding theorem statement, the vector space $\mathscr{V}$ can also be chosen as a space of block-diagonal symmetric matrices, like $\mathscr{S}$, and the above result would still hold. This is particularly convenient for our case. Frequently, one might be tempted to use a subspace of $\mathscr{S}$, rather than the whole space $\mathscr{S}$, since some LMIs might have special structures; this is also appealing because the subspace of a finite-dimensional Hilbert space is itself a Hilbert space, and hence self-dual. But then the result of the above theorem may no longer hold. Specifically, if we are to apply the above theorem to the synthesis conditions (11)–(13), the special structure of the coupling condition (13) might suggest using a subspace of $\mathscr{S}$ instead of $\mathscr{S}$, notably one consisting of block-diagonal symmetric matrices in $\mathscr{S}$ where a number of the matrix blocks, namely those corresponding to the coupling condition, are of the following form:

$$\begin{bmatrix} X & \alpha I \\ \alpha I & Y \end{bmatrix}$$

with X, Y being symmetric matrices, and $\alpha \in \mathbb{R}$. But, in such a case, it is not difficult to show by a counter example that the result of Theorem 11 no longer applies.

The next result follows directly from Theorem 11.

Proposition 13
Suppose that the operators P, Q, and T are block-diagonal and (h, q)-eventually periodic, and that $T = T^*$. Then, given an integer $N \geqslant h$, exactly one of the following statements is true:

(i) There exists an (N, q)-eventually periodic operator $X \in \mathscr{X}_g$ satisfying inequality (15);
(ii) There exists a block-diagonal matrix $Y = \mathrm{diag}(Y_0, Y_1, \ldots, Y_{N+q-1}) \neq 0$, where $Y_i \succcurlyeq 0$ is of the same dimension as the square matrix $Q_i^* Q_i$ for all i, such that

$$\begin{gathered} \mathrm{trace}(\tilde{T} Y) \geqslant 0 \\ \varepsilon_1 (Z_1 \tilde{P} Y \tilde{P}^* Z_1^* + Z_2 \tilde{P} Y \tilde{P}^* Z_2^*) + \varepsilon_2 \tilde{Q} Y \tilde{Q}^* \succcurlyeq 0 \end{gathered} \tag{17}$$

Proof
To start, we define the following sets:

$$\mathscr{V} = \{\tilde{X} = \operatorname{diag}(X_0, X_1, \ldots, X_{N+q-1}) : X_i = X_i^* \quad \text{and} \quad \dim(X_i) = \dim(Q_i Q_i^*)\}$$

$$\mathscr{S} = \{\Gamma = \operatorname{diag}(\Gamma_0, \Gamma_1, \ldots, \Gamma_{N+q-1}, \tilde{X}) : \tilde{X} \in \mathscr{V}, \Gamma_i = \Gamma_i^* \quad \text{and} \quad \dim(\Gamma_i) = \dim(Q_i^* Q_i)\}$$

Also, we define the linear map $\mathfrak{E} : \mathscr{V} \to \mathscr{S}$ by

$$\mathfrak{E}(\tilde{X}) = \operatorname{diag}(-\varepsilon_1 \tilde{P}^* Z_1^* \tilde{X} Z_1 \tilde{P} - \varepsilon_1 \tilde{P}^* Z_2^* \tilde{X} Z_2 \tilde{P} - \varepsilon_2 \tilde{Q}^* \tilde{X} \tilde{Q}, \tilde{X})$$

Then, setting $E_0 = \operatorname{diag}(-\tilde{T}, 0) \in \mathscr{S}$, we can equivalently write inequality (15), together with the constraint $\tilde{X} \succ 0$, as $\mathfrak{E}(\tilde{X}) + E_0 \succ 0$.

Lastly, we define the adjoint of the map $\mathfrak{E}$ by $\mathfrak{E}^* : \mathscr{S} \to \mathscr{V}$ such that for all matrices $\tilde{X} \in \mathscr{V}$ and $\bar{Y} = \operatorname{diag}(Y, \check{Y}) \in \mathscr{S}$, where $\check{Y} \in \mathscr{V}$, we have

$$\langle \mathfrak{E}(\tilde{X}), \bar{Y} \rangle_{\mathscr{S}} = \langle \tilde{X}, \mathfrak{E}^*(\bar{Y}) \rangle_{\mathscr{V}}$$

with $\langle A, B \rangle = \operatorname{trace}(AB)$. It can be easily verified that $\mathfrak{E}^* : \mathscr{S} \to \mathscr{V}$ is given by

$$\mathfrak{E}^*(\bar{Y}) = -\varepsilon_1 Z_1 \tilde{P} Y \tilde{P}^* Z_1^* - \varepsilon_1 Z_2 \tilde{P} Y \tilde{P}^* Z_2^* - \varepsilon_2 \tilde{Q} Y \tilde{Q}^* + \check{Y}$$

Then, by invoking Theorem 11, we deduce that the non-existence of a matrix $\tilde{X} \in \mathscr{V}$ satisfying inequality $\mathfrak{E}(\tilde{X}) + E_0 \succ 0$ is equivalent to the existence of a non-zero matrix $\bar{Y} \in \mathscr{S}$ such that $\bar{Y} \succcurlyeq 0$, $\mathfrak{E}^*(\bar{Y}) = 0$, and $\langle E_0, \bar{Y} \rangle_{\mathscr{S}} \leqslant 0$, which is, in turn, clearly equivalent to statement (ii). Note that the non-zero constraint ($\bar{Y} \neq 0$) can be restricted to the matrix Y since, if $Y = 0$, then the equality $\mathfrak{E}^*(\bar{Y}) = 0$ implies that $\bar{Y} = 0$. □

Following is an alternative statement of Theorem 6.

Theorem 14
Suppose that integer $N \geqslant h$. There exists *no* admissible (N, q)-eventually periodic synthesis for (h, q)-eventually periodic plant G if and only if there exist block-diagonal matrices $Y_f \succcurlyeq 0$, $Y_b \succcurlyeq 0$, and Y_c of the form $Y_j = \operatorname{diag}(Y_{j,0}, Y_{j,1}, \ldots, Y_{j,N+q-1})$ for $j = f, b, c$, where $\dim(Y_{f,i}) = \dim(F_i^* F_i)$, $\dim(Y_{b,i}) = \dim(J_i^* J_i)$, and $Y_{ci} \in \mathbb{R}^{n_i \times n_i}$, such that $(Y_f, Y_b) \neq (0, 0)$ and

$$\operatorname{trace}(\tilde{H} Y_f) + \operatorname{trace}(\tilde{W} Y_b) \geqslant 2\operatorname{trace}(Y_c)$$

$$\begin{bmatrix} -\tilde{F} Y_f \tilde{F}^* + Z_1 \tilde{V}_1 Y_f \tilde{V}_1^* Z_1^* + Z_2 \tilde{V}_1 Y_f \tilde{V}_1^* Z_2^* & Y_c \\ Y_c^* & -Z_1 \tilde{J} Y_b \tilde{J}^* Z_1^* - Z_2 \tilde{J} Y_b \tilde{J}^* Z_2^* + \tilde{U}_1 Y_b \tilde{U}_1^* \end{bmatrix} \preccurlyeq 0 \tag{18}$$

This result stems from Theorems 6 and 11. Note that, at a first glance, the fact that the coupling condition (13) is a non-strict inequality seems to undermine the applicability of Theorem 11. However, because of the continuity property of LMIs, a slight perturbation of the solutions ensures that the existence of solutions in $\tilde{\mathscr{X}}$ to the synthesis conditions of Theorem 6 is equivalent to the existence of matrices $\tilde{R}, \tilde{S}, \in \tilde{\mathscr{X}}$ satisfying LMIs (11) and (12), respectively, as well as the strict coupling inequality

$$\begin{bmatrix} \tilde{R} & I \\ I & \tilde{S} \end{bmatrix} \succ 0$$

Then, Theorem 11 is clearly applicable, and so following a similar argument to that of the proof of Proposition 13, we can prove Theorem 14; the complete proof is omitted for space considerations. Note that the coupling condition (13) already implies that $\tilde{R}, \tilde{S} \succ 0$, and so we need not redundantly account for the positive definiteness of the solutions.

In the following, we will say that inequality (14) is a *primal* LMI and that inequalities (17) are the corresponding *alternative* LMIs; similarly, inequalities (18) are the alternative LMIs for the primal synthesis conditions (11)–(13). Appealing to Proposition 7 and Theorem 8, it is clear that the periodicity of any eventually periodic solution in the context of these results is always equal to that of the plant, namely q; the only variable in general is the finite horizon length of such a solution. With that in mind, and for notational clarity and simplicity, we will denote the solutions of the alternative LMIs only in terms of the finite horizon lengths of the primal eventually q-periodic solutions as follows.

Definition 15

A matrix Y is said to be an N-solution of alternative LMIs (17) if Y is a *non-zero* block-diagonal matrix of the form $Y = \mathrm{diag}(Y_0, Y_1, \ldots, Y_{N+q-1})$, where $Y_i \succcurlyeq 0$ is of the same dimension as the square matrix $Q_i^* Q_i$ for all i. Furthermore, the triplet (Y_f, Y_b, Y_c) is said to be an N-solution of alternative LMIs (18) if $(Y_f, Y_b) \neq (0, 0)$, and $Y_f \succcurlyeq 0$, $Y_b \succcurlyeq 0$, and Y_c are block-diagonal matrices of the form $Y_j = \mathrm{diag}(Y_{j,0}, Y_{j,1}, \ldots, Y_{j,N+q-1})$ for $j = f, b, c$, where $\dim(Y_{f,i}) = \dim(F_i^* F_i)$, $\dim(Y_{b,i}) = \dim(J_i^* J_i)$, and $Y_{ci} \in \mathbb{R}^{n_i \times n_i}$.

Clearly, the existence of an N-solution to some alternative LMIs implies the existence of a τ-solution for all integers τ, where $h \leqslant \tau \leqslant N$.

6. KYP LEMMA FOR EVENTUALLY PERIODIC SYSTEMS

This section gives an alternative derivation of an important version of the KYP lemma for eventually periodic systems, originally given in Reference [1]. Specifically, we will utilize Propositions 7 and 13 to show that, given an (N, q)-eventually periodic closed-loop system, there exists a solution in $\mathscr{X}^L$ to inequality (5) if and only if there exists an (N, q)-eventually periodic solution in $\mathscr{X}^L$.

Relating inequality (14) to (5), we set $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, and further make the following assignments for all integers $i \geqslant 0$:

$$P_i = [A_i^L \quad B_i^L], \quad Q_i = [I \quad 0], \quad T_i = \begin{bmatrix} C_i^{L*} C_i^L & C_i^{L*} D_i^L \\ D_i^{L*} C_i^L & D_i^{L*} D_i^L - I \end{bmatrix}$$

Then, we write inequality (5) more conveniently as

$$P^*Z^*XZP - Q^*XQ + T \prec 0 \tag{19}$$

Following are a couple of useful propositions; the proofs are given in the appendix for completeness.

Proposition 16
Suppose that X is a positive semidefinite matrix partitioned as follows:

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$$

where $\operatorname{rank}(X_{11}) = m \geqslant 1$. Then, there exist matrices Σ, Γ and Ω of compatible dimensions such that Σ has full column rank equal to m and

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ \Gamma & \Omega \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ \Gamma & \Omega \end{bmatrix}^* = \begin{bmatrix} \Sigma\Sigma^* & \Sigma\Gamma^* \\ \Gamma\Sigma^* & \Gamma\Gamma^* + \Omega\Omega^* \end{bmatrix}$$

Proposition 17
Given matrices Σ and Γ such that $\Sigma\Sigma^* \preccurlyeq \Gamma\Gamma^*$, then there exists a matrix Ω such that $\Omega^*\Omega \preccurlyeq I$ and $\Gamma\Omega = \Sigma$.

Next we give a new proof of the following result from Reference [1].

Theorem 18
Given an (N, q)-eventually periodic closed-loop system, the existence of a solution in $\mathscr{X}^L$ to inequality (5) is equivalent to the existence of an (N, q)-eventually periodic solution in $\mathscr{X}^L$.

Proof
Note that, out of all the finite horizon, only the last instant is relevant to this proof, and hence we assume without loss of generality that the finite horizon length N is equal to 1; the q-periodic case, where $N = 0$, is already addressed in Part (i) of Proposition 7. Since the converse is immediate, we only need to prove the claim that if a solution in $\mathscr{X}^L$ exists to inequality (19), then a $(1, q)$-eventually periodic solution in $\mathscr{X}^L$ exists. After invoking Part (ii) of Proposition 7, the contrapositive of this claim is as follows: if no $(1, q)$-eventually periodic solution in $\mathscr{X}^L$ exists to inequality (19), then there does not exist any (τ, q)-eventually periodic solution in $\mathscr{X}^L$ for all $\tau \geqslant 1$; or equivalently, by Proposition 13, if there exists a 1-solution satisfying the alternative LMIs to inequality (19), then there exists a τ-solution for all $\tau \geqslant 1$. Proposition 13 entails that the non-existence of a $(1, q)$-eventually periodic solution in $\mathscr{X}^L$ to

inequality (19) is equivalent to the existence of a 1-solution $Y = \text{diag}(Y_0, Y_1, \ldots, Y_q)$ to the following LMIs:

$$\begin{gathered}\textstyle\sum_{i=0}^{q} \text{trace}(T_i Y_i) \geqslant 0 \\ Q_0 Y_0 Q_0^* = 0 \\ Q_1 Y_1 Q_1^* \preccurlyeq P_0 Y_0 P_0^* + P_q Y_q P_q^* \\ Q_2 Y_2 Q_2^* \preccurlyeq P_1 Y_1 P_1^* \\ \vdots \\ Q_q Y_q Q_q^* \preccurlyeq P_{q-1} Y_{q-1} P_{q-1}^*\end{gathered} \tag{20}$$

where $Y \neq 0$, $Y_i \succcurlyeq 0$ is a square matrix of dimension $n_i + r_i + n_{wi}$, and

$$Q_i Y_i Q_i^* = [I_{n_i+r_i} \;\; 0] \begin{bmatrix} Y_{i,11} & Y_{i,12} \\ Y_{i,12}^* & Y_{i,22} \end{bmatrix} \begin{bmatrix} I_{n_i+r_i} \\ 0 \end{bmatrix} = Y_{i,11}$$

Suppose that $Y_{1,11} \neq 0$, then, by Proposition 16, there exist matrices Σ, Γ, and Ω of compatible dimensions such that Σ has full column rank equal to rank$(Y_{1,11})$, and

$$Y_1 = \begin{bmatrix} Y_{1,11} & Y_{1,12} \\ Y_{1,12}^* & Y_{1,22} \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ \Gamma & \Omega \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ \Gamma & \Omega \end{bmatrix}^* = \begin{bmatrix} \Sigma\Sigma^* & \Sigma\Gamma^* \\ \Gamma\Sigma^* & \Gamma\Gamma^* + \Omega\Omega^* \end{bmatrix}$$

In the event that $Y_{1,11} = 0$, then also $Y_{1,12} = 0$ since $Y_1 \succcurlyeq 0$, and the above still applies except that $\Sigma = 0$, $\Gamma = 0$ and $Y_{1,22} = \Omega\Omega^*$. Now, clearly $P_0 Y_0 P_0^* \succcurlyeq 0$ and $P_q Y_q P_q^* \succcurlyeq 0$, and hence, there exist matrices Σ_1 and Σ_2 of compatible dimensions such that $P_0 Y_0 P_0^* = \Sigma_1 \Sigma_1^*$ and $P_q Y_q P_q^* = \Sigma_2 \Sigma_2^*$. Then the inequality $Q_1 Y_1 Q_1^* \preccurlyeq P_0 Y_0 P_0^* + P_q Y_q P_q^*$ can be equivalently written as $\Sigma\Sigma^* \preccurlyeq [\Sigma_1 \;\; \Sigma_2][\Sigma_1 \;\; \Sigma_2]^*$, and so by Proposition 17, there exists a matrix Θ such that $\Theta^*\Theta \preccurlyeq I$ and $[\Sigma_1 \;\; \Sigma_2]\Theta = \Sigma$. At this point, we partition the matrix $\Gamma\Theta^* = [\Gamma_1 \;\; \Gamma_2]$ in accordance with the partitioning of the matrix $[\Sigma_1 \;\; \Sigma_2]$, and further define the matrix $\Psi = I - \Theta^*\Theta \succcurlyeq 0$ and

$$\bar{Y}_1 = \begin{bmatrix} \Sigma_1 & 0 \\ \Gamma_1 & \Omega \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ \Gamma_1 & \Omega \end{bmatrix}^* = \begin{bmatrix} \Sigma_1\Sigma_1^* & \Sigma_1\Gamma_1^* \\ \Gamma_1\Sigma_1^* & \Gamma_1\Gamma_1^* + \Omega\Omega^* \end{bmatrix} \succcurlyeq 0$$

$$\bar{Y}_i = \begin{bmatrix} P_{i-1} Y_{i-1} P_{i-1}^* & Y_{i,12} \\ Y_{i,12}^* & Y_{i,22} \end{bmatrix} \succcurlyeq 0 \quad \text{for } i = 2, 3, \ldots, q$$

$$\bar{Y}_{q+1} = \begin{bmatrix} \Sigma_2 & 0 \\ \Gamma_2 & \Gamma\Psi^{1/2} \end{bmatrix} \begin{bmatrix} \Sigma_2 & 0 \\ \Gamma_2 & \Gamma\Psi^{1/2} \end{bmatrix}^* = \begin{bmatrix} \Sigma_2\Sigma_2^* & \Sigma_2\Gamma_2^* \\ \Gamma_2\Sigma_2^* & \Gamma_2\Gamma_2^* + \Gamma\Psi\Gamma^* \end{bmatrix} \succcurlyeq 0$$

Note that $\bar{Y}_i \succcurlyeq 0$ for $i = 2, \ldots, q$ since, from (20), $P_{i-1} Y_{i-1} P_{i-1}^* \succcurlyeq Y_{i,11}$. It can be easily verified that the following LMIs hold:

$$\begin{gathered}
\text{trace}(T_0 Y_0) + \textstyle\sum_{i=1}^{q+1} \text{trace}(T_i \bar{Y}_i) \geqslant 0 \\
Q_0 Y_0 Q_0^* = 0 \\
Q_1 \bar{Y}_1 Q_1^* \preccurlyeq P_0 Y_0 P_0^* \\
Q_2 \bar{Y}_2 Q_2^* \preccurlyeq P_1 \bar{Y}_1 P_1^* + P_1 \bar{Y}_{q+1} P_1^* \\
Q_3 \bar{Y}_3 Q_3^* \preccurlyeq P_2 \bar{Y}_2 P_2^* \\
\vdots \\
Q_{q+1} \bar{Y}_{q+1} Q_{q+1}^* = Q_1 \bar{Y}_{q+1} Q_1^* \preccurlyeq P_q \bar{Y}_q P_q^*
\end{gathered}$$

where $\bar{Y} = \text{diag}(Y_0, \bar{Y}_1, \ldots, \bar{Y}_{q+1}) \neq 0$. Thus, we have constructed a 2-solution $\bar{Y}$ to the alternative LMIs from a given 1-solution. Finally, we can apply the previous argument recursively, and show that, given a 1-solution to (20), we can always construct a τ-solution for all $\tau \geqslant 1$. □

7. NEW SYNTHESIS RESULTS

As stated in Theorem 8, given an (h, q)-eventually periodic plant, the existence of an admissible synthesis for this plant is equivalent to the existence of an (N, q)-eventually periodic synthesis for some $N \geqslant h$. While this result establishes that the solution of the synthesis ℓ_2-induced control problem for eventually periodic systems can always be expressed in terms of *finite-dimensional* convex conditions, it does not exactly specify these conditions. Granted that such conditions are somewhat truncations of the synthesis conditions (6–8), yet the extent of such truncations is not explicit in the theorem statement. In fact, the value of N above is still not known in general. This section is divided into two subsections. The first one closely examines each of the synthesis conditions (6), (7) and (8) to see which of these conditions contribute to the indefiniteness of N. Then, based on our findings, we consider in the second subsection two cases where an admissible synthesis exists if and only if an admissible (h, q)-eventually periodic synthesis exists.

7.1. Synthesis conditions

We start with the following theorem pertaining to the synthesis condition (6).

Theorem 19
Given an (h, q)-eventually periodic plant, then a solution in $\mathscr{X}$ exists to the synthesis condition (6) if and only if an (h, q)-eventually periodic solution in $\mathscr{X}$ exists.

Proof
As in the proof of Theorem 18, we will assume without loss of generality that the finite horizon length h is equal to 1. The proof of the 'if' direction is immediate, and so we only need to prove the 'only if' direction. Hence, we need to show that the existence of a 1-solution to the

alternative LMIs of inequality (11), as given by Proposition 13, implies the existence of an N-solution for all $N \geqslant 1$. Suppose that the non-zero matrix $Y = \mathrm{diag}(Y_0, Y_1, \ldots, Y_q)$ is a 1-solution of the aforementioned alternative LMIs, namely:

$$\begin{gathered}
\textstyle\sum_{i=0}^{q} \mathrm{trace}(H_i Y_i) \geqslant 0 \\
V_{10} Y_0 V_{10}^* + V_{1q} Y_q V_{1q}^* \preccurlyeq F_1 Y_1 F_1^* \\
V_{11} Y_1 V_{11}^* \preccurlyeq F_2 Y_2 F_2^* \\
\vdots \\
V_{1,q-1} Y_{q-1} V_{1,q-1}^* \preccurlyeq F_q Y_q F_q^*
\end{gathered}$$

The second LMI in the above sequence can be equivalently written as

$$[V_{10} Y_0^{1/2} \;\; V_{1q} Y_q^{1/2}][V_{10} Y_0^{1/2} \;\; V_{1q} Y_q^{1/2}]^* \preccurlyeq F_1 Y_1 F_1^*$$

and hence, by Proposition 17, there exists a matrix Σ such that $\Sigma\Sigma^* \preccurlyeq I$, and

$$F_1 Y_1^{1/2} \Sigma = [V_{10} Y_0^{1/2} \;\; V_{1q} Y_q^{1/2}]$$

Partition Σ appropriately as $\Sigma = [\Sigma_1 \;\; \Sigma_2]$ so that

$$F_1 Y_1^{1/2} \Sigma_1 = V_{10} Y_0^{1/2} \quad \text{and} \quad F_1 Y_1^{1/2} \Sigma_2 = V_{1q} Y_q^{1/2}$$

Then, for any Γ of proper dimensions such that $\Gamma\Gamma^* \preccurlyeq I$, define the following matrices:

$$\begin{gathered}
Y_{1a} = Y_1^{1/2} (\Sigma_1 \Sigma_1^* + (I - \Sigma\Sigma^*)^{1/2} (I - \Gamma\Gamma^*) (I - \Sigma\Sigma^*)^{1/2}) Y_1^{1/2} \\
Y_{1b} = Y_1^{1/2} (\Sigma_2 \Sigma_2^* + (I - \Sigma\Sigma^*)^{1/2} \Gamma\Gamma^* (I - \Sigma\Sigma^*)^{1/2}) Y_1^{1/2}
\end{gathered}$$

Note that $Y_{1a} \succcurlyeq 0$, $Y_{1b} \succcurlyeq 0$ and $Y_{1a} + Y_{1b} = Y_1$. Then the following LMIs clearly hold:

$$\begin{gathered}
\mathrm{trace}(H_0 Y_0) + \mathrm{trace}(H_1 Y_{1a}) + \textstyle\sum_{i=2}^{q} \mathrm{trace}(H_i Y_i) + \mathrm{trace}(H_1 Y_{1b}) \geqslant 0 \\
V_{10} Y_0 V_{10}^* \preccurlyeq F_1 Y_{1a} F_1^* \\
V_{11} Y_1 V_{11}^* = V_{11} Y_{1a} V_{11}^* + V_{11} Y_{1b} V_{11}^* \preccurlyeq F_2 Y_2 F_2^* \\
V_{12} Y_2 V_{12}^* \preccurlyeq F_3 Y_3 F_3^* \\
\vdots \\
V_{1,q-1} Y_{q-1} V_{1,q-1}^* \preccurlyeq F_q Y_q F_q^* \\
V_{1q} Y_q V_{1q}^* \preccurlyeq F_1 Y_{1b} F_1^*
\end{gathered}$$

Thus, the matrix $\mathrm{diag}(Y_0, Y_{1a}, Y_2, \ldots, Y_q, Y_{1b})$ is a 2-solution to the alternative LMIs of inequality (11). So, we have constructed a 2-solution from a given 1-solution. Finally, we can apply the previous argument recursively, and show that, given a 1-solution to the alternative LMIs, we can always construct an N-solution for all $N \geqslant 1$. □

Note that a different proof which does not use any duality results is given in the conference paper [3]; the proof above though is far simpler and more concise.

The result of Theorem 19 does not apply in general to the synthesis condition (7). It is not difficult to construct counter examples to show that, given an (h, q)-eventually periodic plant, the existence of a solution in $\mathscr{X}$ to inequality (7) does not necessarily imply the existence of an (h, q)-eventually periodic solution in $\mathscr{X}$. However, if we are to drop the constraint that the solution has to be positive definite, and expand our search for solutions to the set $\mathscr{X}_e$, whose elements are bounded, self-adjoint and of the same form as those of the set $\mathscr{X}$, but without the positive definiteness restriction, then we have the following result.

Theorem 20
Given an (h, q)-eventually periodic plant, there exists a solution in $\mathscr{X}_e$ to the synthesis condition (7) if and only if there exists an (h, q)-eventually periodic solution in $\mathscr{X}_e$.

Proof
Note that the proofs of Proposition 7 can be easily rephrased to suit this case, and as a result, a solution in $\mathscr{X}_e$ exists to inequality (7) if and only if an eventually q-periodic solution in $\mathscr{X}_e$ exists. As in the proof of Theorem 18, we will assume without loss of generality that the finite horizon length h is equal to 1. Since the converse is immediate, we only need to prove the 'only if' direction of the above result. To start, we have to alter Proposition 13 slightly so as to apply in this case where the primal solution belongs to the set $\mathscr{X}_e$ instead of $\mathscr{X}$. By appealing to the proof of this proposition, it is not difficult to see that in this case the alternative LMIs (17) change as follows. The first LMI, i.e. the trace condition, stays the same, while the second LMI becomes an equality; the solution to the alternative LMIs still has to be non-zero, positive semidefinite and block-diagonal. With this in mind, and while still adopting the same notation as before for simplicity, we need to show that, given a 1-solution to the alternative LMIs, then there exists an N-solution for all $N \geqslant 1$. Suppose that the non-zero matrix $Y = \text{diag}(Y_0, Y_1, \ldots, Y_q) \succcurlyeq 0$ is a 1-solution to the alternative LMIs; hence the following hold:

$$\textstyle\sum_{i=0}^{q} \text{trace}(W_i Y_i) \geqslant 0$$

$$U_{10} Y_0 U_{10}^* = 0$$

$$U_{11} Y_1 U_{11}^* = J_0 Y_0 J_0^* + J_q Y_q J_q^*$$

$$U_{12} Y_2 U_{12}^* = J_1 Y_1 J_1^*$$

$$\vdots$$

$$U_{1q} Y_q U_{1q}^* = J_{q-1} Y_{q-1} J_{q-1}^*$$

Focusing on the equality $U_{11} Y_1 U_{11}^* = J_0 Y_0 J_0^* + J_q Y_q J_q^*$, we can always find matrices Θ_1 and Θ_2 such that

$$\Theta_1 \Theta_1^* = J_0 Y_0 J_0^*, \quad \Theta_2 \Theta_2^* = J_q Y_q J_q^*, \quad \text{and } \dim([\Theta_1 \ \ \Theta_2]^* [\Theta_1 \ \ \Theta_2]) \geqslant \dim(Y_1)$$

Then, it is immediate that there exists a matrix Σ of appropriate dimensions such that $\Sigma\Sigma^* = I$ and $U_{11} Y_1^{1/2} \Sigma = [\Theta_1 \ \ \Theta_2]$. Partition $\Sigma = [\Sigma_1 \ \ \Sigma_2]$ conformably with the partitioning of the matrix $[\Theta_1 \ \ \Theta_2]$ so that

$$U_{11} Y_1^{1/2} \Sigma_1 = \Theta_1 \quad \text{and} \quad U_{11} Y_1^{1/2} \Sigma_2 = \Theta_2$$

Define $Y_{1a} = Y_1^{1/2}\Sigma_1\Sigma_1^* Y_1^{1/2} \succcurlyeq 0$, and $Y_{1b} = Y_1^{1/2}\Sigma_2\Sigma_2^* Y_1^{1/2} \succcurlyeq 0$; clearly, $Y_{1a} + Y_{1b} = Y_1$. Consequently, we have

$$\begin{gathered}
\operatorname{trace}(W_0 Y_0) + \operatorname{trace}(W_1 Y_{1a}) + \textstyle\sum_{i=2}^{q} \operatorname{trace}(W_i Y_i) + \operatorname{trace}(W_1 Y_{1b}) \geqslant 0 \\
U_{10} Y_0 U_{10}^* = 0 \\
U_{11} Y_{1a} U_{11}^* = J_0 Y_0 J_0^* \\
U_{12} Y_2 U_{12}^* = J_1 Y_{1a} J_1^* + J_1 Y_{1b} J_1^* \\
U_{13} Y_3 U_{13}^* = J_2 Y_2 J_2^* \\
\vdots \\
U_{1q} Y_q U_{1q}^* = J_{q-1} Y_{q-1} J_{q-1}^* \\
U_{11} Y_{1b} U_{11}^* = J_q Y_q J_q^*
\end{gathered}$$

Thus, the matrix $\operatorname{diag}(Y_0, Y_{1a}, Y_2, \ldots, Y_q, Y_{1b})$ is a 2-solution to the alternative LMIs, constructed from the given 1-solution Y. Finally, we can apply the previous argument recursively, and show that, given a 1-solution to the alternative LMIs, we can always construct an N-solution for all $N \geqslant 1$. □

Remark 21

Another proof to Theorem 20 which does not employ any theorems of alternatives is outlined as follows. First, by Finsler's lemma, the existence of a solution in $\mathscr{X}_e$ to the synthesis condition (7) is equivalent to the existence of an operator $S \in \mathscr{X}_e$ and a scalar $\alpha > 0$ such that

$$\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* \begin{bmatrix} Z^*SZ & \\ & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - \begin{bmatrix} S & \\ & I \end{bmatrix} - \alpha \begin{bmatrix} C_2^* \\ D_{21}^* \end{bmatrix} [C_2 \ \ D_{21}] \prec 0 \tag{21}$$

Formally define

$$\Omega_i(X) := A_i^* X A_i + C_{1i}^* C_{1i} - \alpha C_{2i}^* C_{2i} - E(B_{1i}^* X B_{1i} + D_{11i}^* D_{11i} - I - \alpha D_{21i}^* D_{21i})^{-1} E^*$$

where $E = A_i^* X B_{1i} + C_{1i}^* D_{11i} - \alpha C_{2i}^* D_{21i}$, so that, by applying the Schur complement formula to (21), we get $\Omega_i(S_{i+1}) \prec S_i$ for all $i \geqslant 0$. Then, following a similar argument to that used for the proof of Theorem 12 in Reference [1], we can construct from any eventually q-periodic operator in $\mathscr{X}_e$ satisfying inequality (7) an (h, q)-eventually periodic solution in $\mathscr{X}_e$. The detailed proof above though is clearly far simpler and more concise.

The preceding theorem is appealing; however, the positive definiteness of the solution of the synthesis condition (7) is necessary for the validity of the coupling condition (8). In fact, it is not difficult to construct examples to show that, even when both of the synthesis conditions (6) and (7) admit solutions in the subclass of the (N, q)-eventually periodic operators of $\mathscr{X}$ for some $N \geqslant h$, none of such solutions might satisfy the coupling condition (8), and accordingly we may need to settle for a larger finite horizon length.

7.2. Special cases

This subsection considers two cases of (h, q)-eventually periodic plants where the synthesis condition (7) simplifies significantly and, as a result, an admissible synthesis if existent can always be chosen to be (h, q)-eventually periodic.

Theorem 22

Suppose that plant G is (h, q)-eventually periodic, and that the periodic part of G has exactly measurable states, that is, for all $i = h, \ldots, h+q-1$, we have $C_{2i} = I$, $D_{21i} = 0$. Then there exists an admissible synthesis K for G, with state dimension $r_i \leqslant n_i$ for all i, if and only if there exists an admissible (h, q)-eventually periodic synthesis.

Proof

As in the previous proofs, assume $h = 1$. Then, we only need to show that the existence of a 1-solution to the alternative LMIs (18) implies the existence of an N-solution for all $N \geqslant 1$. Note that, in this case, appealing to the definitions in (9), it is not difficult to see that $U_{1i} = 0$, $U_{2i} = I$ for all $i = 1, 2, \ldots, q$. If we start with a 2-solution, then the argument for constructing from this solution an N-solution for all $N \geqslant 2$ is rather simple, as we will show next. On the other hand, constructing a 2-solution from a 1-solution is slightly more involved and is accordingly provided in the appendix. We start with a 2-solution (Y_f, Y_b, Y_c) to the alternative LMIs (18); hence the following hold:

$$\sum_{i=0}^{q+1} \text{trace}(H_i Y_{fi}) + \sum_{i=0}^{q+1} \text{trace}(W_i Y_{bi}) \geqslant 2 \sum_{i=1}^{q+1} \text{trace}(Y_{ci})$$

$$U_{10} Y_{b0} U_{10}^* = 0$$

$$\begin{bmatrix} V_{10} Y_{f0} V_{10}^* - F_1 Y_{f1} F_1^* & Y_{c1} \\ Y_{c1}^* & -B_{10} Y_{b0} B_{10}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{11} Y_{f1} V_{11}^* + V_{11} Y_{f,q+1} V_{11}^* - F_2 Y_{f2} F_2^* & Y_{c2} \\ Y_{c2}^* & -B_{11} Y_{b1} B_{11}^* - B_{11} Y_{b,q+1} B_{11}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{12} Y_{f2} V_{12}^* - F_3 Y_{f3} F_3^* & Y_{c3} \\ Y_{c3}^* & -B_{12} Y_{b2} B_{12}^* \end{bmatrix} \preccurlyeq 0$$

$$\vdots$$

$$\begin{bmatrix} V_{1,q-1} Y_{f,q-1} V_{1,q-1}^* - F_q Y_{fq} F_q^* & Y_{cq} \\ Y_{cq}^* & -B_{1,q-1} Y_{b,q-1} B_{1,q-1}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{1q} Y_{fq} V_{1q}^* - F_1 Y_{f,q+1} F_1^* & Y_{c,q+1} \\ Y_{c,q+1}^* & -B_{1q} Y_{bq} B_{1q}^* \end{bmatrix} \preccurlyeq 0$$

The fourth LMI in the preceding sequence implies that $V_{11} Y_{f1} V_{11}^* + V_{11} Y_{f,q+1} V_{11}^* \preccurlyeq F_2 Y_{f2} F_2^*$. Then appealing to the argument in the proof of Theorem 19, there exists a matrix $\Sigma = [\Sigma_1 \ \ \Sigma_2]$ such that

$$\Sigma\Sigma^* \preccurlyeq I, \ \ F_2 Y_{f2}^{1/2} \Sigma_1 = V_{11} Y_{f1}^{1/2}, \quad \text{and} \quad F_2 Y_{f2}^{1/2} \Sigma_2 = V_{11} Y_{f,q+1}^{1/2}$$

Then we define $Y_{f2,a} = Y_{f2}^{1/2}\Sigma_1\Sigma_1^* Y_{f2}^{1/2} \succcurlyeq 0$ and $Y_{f2,b} = Y_{f2}^{1/2}(I - \Sigma_1\Sigma_1^*)Y_{f2}^{1/2} \succcurlyeq 0$. The following ensue: $Y_{f2,a} + Y_{f2,b} = Y_{f2}$, $V_{11}Y_{f1}V_{11}^* = F_2Y_{f2,a}F_2^*$, and

$$V_{11}Y_{f,q+1}V_{11}^* - F_2Y_{f2,b}F_2^* = V_{11}Y_{f1}V_{11}^* + V_{11}Y_{f,q+1}V_{11}^* - F_2Y_{f2}F_2^*$$

Defining $\bar{Y}_f = \operatorname{diag}(Y_{f0}, Y_{f1}, Y_{f2,a}, Y_{f3}, \ldots, Y_{f,q+1}, Y_{f2,b})$, $\bar{Y}_b = \operatorname{diag}(Y_{b0}, 0, Y_{b2}, \ldots, Y_{bq}, Y_{b1} + Y_{b,q+1}, 0)$, and $\bar{Y}_c = \operatorname{diag}(0, Y_{c1}, 0, Y_{c3}, \ldots, Y_{c,q+1}, Y_{c1})$, then the triplet $(\bar{Y}_f, \bar{Y}_b, \bar{Y}_c)$ constitutes a 3-solution to the alternative LMIs (18). Following the same argument recursively, we can construct from any 2-solution to the alternative LMIs an N-solution for all $N \geqslant 2$. This provisionally ends the proof of Theorem 22. □

Remark 23

In the event that $C_{2i} = I$ and $D_{21i} = 0$ for all $i = 0, 1, \ldots, h+q-1$, that is the (h,q)-eventually periodic plant G has exactly measurable states, then an admissible synthesis for G exists if and only if an admissible (h,q)-eventually periodic *static* synthesis exists, as shown in Theorem 14 of Reference [3]. Also, for such a case, instead of first trying to find solutions to the synthesis conditions and then, if successful, solving for the static controller, we may lump both of these steps into one, as shown in Theorem 15 of Reference [3]. As for the more general case where only the periodic part of plant G has exactly measurable states, it is not difficult to generalize the proof of Theorem 14 of Reference [3] to show that an admissible synthesis if existent can always be chosen to be (h,q)-eventually periodic such that the periodic part is *static*, that is the control law $u_i = D_i^K y_i$ for $i = h, h+1, \ldots$; note that, in this case, A_{h-1}^K and B_{h-1}^K are empty matrices with zero row dimensions.

Theorem 24

Suppose that plant G is (h,q)-eventually periodic, and that the state disturbance is a linear transformation of the sensor noise at each point of the finite horizon, that is, for all $i = 0, 1, \ldots, h-1$, we have $B_{1i} = T_iD_{21i}$ for some matrix T_i. Then there exists an admissible synthesis K for G, with state dimension $r_i \leqslant n_i$ for all i, if and only if there exists an admissible (h,q)-eventually periodic synthesis.

Proof

We only need to prove the 'only if' direction. Consider an h-solution (Y_f, Y_b, Y_c) to the alternative LMIs (18). Then the following are necessarily valid:

$$\begin{aligned} U_{10}Y_{b0}U_{10}^* &= 0 \\ U_{1,i+1}Y_{b,i+1}U_{1,i+1}^* &\preccurlyeq J_iY_{bi}J_i^* \end{aligned} \tag{22}$$

for $i = 0, 1, \ldots, h-2$. From the definitions in (9), we have $C_{2i}U_{1i} + D_{21i}U_{2i} = 0$ for all $i \geqslant 0$, and since in this case $B_{1i} = T_iD_{21i}$ for some matrix T_i at each instant in the finite horizon, then clearly the equality $T_iC_{2i}U_{1i} = -B_{1i}U_{2i}$ holds for all $i = 0, 1, \ldots, h-1$. Accordingly, as the matrix $U_{10}Y_{b0}U_{10}^* = 0$, then $B_{10}U_{20}Y_{b0}U_{20}^*B_{10}^* = J_0Y_{b0}J_0^* = 0$, and so, from (22), we get $U_{11}Y_{b1}U_{11}^* = 0$, and so on. Consequently, we have $U_{1i}Y_{bi}U_{1i}^* = 0$ and $J_iY_{bi}J_i^* = 0$ for $i = 0, 1, \ldots, h-1$; the first equality implies that $\operatorname{trace}(W_iY_{bi}) \leqslant 0$ and, together with the second one, justifies taking $Y_{bi} = 0$ for $i = 0, 1, \ldots, h-1$. Having mentioned this, we assume $h = 1$ as this clearly does not detract from the generality of the ensuing proof. So now, the triplet (Y_f, Y_b, Y_c)

is a 1-solution to the LMIs (18) with $Y_{b0} = 0$, namely,

$$\sum_{i=0}^{q} \text{trace}(H_i Y_{fi}) + \sum_{i=1}^{q} \text{trace}(W_i Y_{bi}) \geqslant 2 \sum_{i=1}^{q} \text{trace}(Y_{ci})$$

$$\begin{bmatrix} V_{10} Y_{f0} V_{10}^* + V_{1q} Y_{fq} V_{1q}^* - F_1 Y_{f1} F_1^* & Y_{c1} \\ Y_{c1}^* & U_{11} Y_{b1} U_{11}^* - J_0(0) J_0^* - J_q Y_{bq} J_q^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{11} Y_{f1} V_{11}^* - F_2 Y_{f2} F_2^* & Y_{c2} \\ Y_{c2}^* & U_{12} Y_{b2} U_{12}^* - J_1 Y_{b1} J_1^* \end{bmatrix} \preccurlyeq 0$$

$$\vdots$$

$$\begin{bmatrix} V_{1,q-1} Y_{f,q-1} V_{1,q-1}^* - F_q Y_{fq} F_q^* & Y_{cq} \\ Y_{cq}^* & U_{1q} Y_{bq} U_{1q}^* - J_{q-1} Y_{b,q-1} J_{q-1}^* \end{bmatrix} \preccurlyeq 0$$

Then, the second inequality in the above sequence of LMIs implies that $V_{10} Y_{f0} V_{10}^* + V_{1q} Y_{fq} V_{1q}^* \preccurlyeq F_1 Y_{f1} F_1^*$. Appealing to the argument in the proof of Theorem 19, there exists a matrix $\Sigma = [\Sigma_1 \;\; \Sigma_2]$ such that

$$\Sigma\Sigma^* \preccurlyeq I, \;\; F_1 Y_{f1}^{1/2} \Sigma_1 = V_{10} Y_{f0}^{1/2}, \quad \text{and} \quad F_1 Y_{f1}^{1/2} \Sigma_2 = V_{1q} Y_{fq}^{1/2}$$

Then defining $Y_{f1,a} = Y_{f1}^{1/2} \Sigma_1 \Sigma_1^* Y_{f1}^{1/2} \succcurlyeq 0$ and $Y_{f1,b} = Y_{f1}^{1/2} (I - \Sigma_1 \Sigma_1^*) Y_{f1}^{1/2} \succcurlyeq 0$, it is clear that the triplet $(\bar{Y}_f, \bar{Y}_b, \bar{Y}_c)$ is a 2-solution to the alternative LMIs (18), where $\bar{Y}_f = \text{diag}(Y_{f0}, Y_{f1,a}, Y_{f2}, \ldots, Y_{fq}, Y_{f1,b})$, $\bar{Y}_b = \text{diag}(0, 0, Y_{b2}, Y_{b3}, \ldots, Y_{bq}, Y_{b1})$, and $\bar{Y}_c = \text{diag}(0, 0, Y_{c2}, Y_{c3}, \ldots, Y_{cq}, Y_{c1})$. Following the same argument recursively, we can construct an N-solution for all $N \geqslant 1$. □

Remark 25

Following are some comments on the last result. Suppose the existence of an admissible synthesis for an (h, q)-eventually periodic plant. If the matrices D_{21i}, for $i = 0, 1, \ldots, h-1$, have each full column rank, then the condition $B_{1i} = T_i D_{21i}$ becomes trivial, and an admissible (h, q)-eventually periodic synthesis exists. Also, the case where the finite horizon matrices U_{1i} have each full column rank guarantees admissible (h, q)-eventually periodic syntheses.

8. CONCLUSIONS

The main contribution of this paper is a new theoretical insight for the analysis and synthesis ℓ_2-induced control problems of eventually periodic systems. The paper shows that, for such systems, the analysis and synthesis solutions can always be provided in terms of finite-dimensional semidefinite programming problems. As there exists a vast literature on duality for such problems, one can utilize various duality results such as theorems of alternatives to better understand the control problem at hand and consequently give new results. In this respect, this paper serves as a gateway for using semidefinite programming duality results in control problems involving eventually periodic systems.

Specifically, we utilize herein a theorem of strong alternatives to give a new proof of an existing analysis result, namely an important version of the KYP lemma for eventually periodic systems, and further closely study the synthesis conditions for the ℓ_2-induced control of these systems. Based on this, we consider two special cases, where the synthesis if existent can always be chosen to be of the same eventually periodic class as the plant.

APPENDIX A

Proof of Proposition 16
By the Schur complement formula, the condition $X \succcurlyeq 0$ is equivalent to

$$X_{11} \succcurlyeq 0, \quad X_{12}^*(I - X_{11}X_{11}^\dagger) = 0, \quad X_{22} - X_{12}^* X_{11}^\dagger X_{12} \succcurlyeq 0 \tag{A1}$$

where $X_{11}^\dagger$ denotes the Moore–Penrose inverse of X_{11}. Since $X_{11} \succcurlyeq 0$, then there exists a matrix Σ with full column rank equal to m such that $X_{11} = \Sigma\Sigma^*$. Also, from the second inequality in (A1), we have $X_{12} = (X_{11}X_{11}^\dagger)^* X_{12} = X_{11}X_{11}^\dagger X_{12} = \Sigma\Gamma^*$, where $\Gamma = X_{12}^* X_{11}^\dagger \Sigma$. Finally, as $X_{12}^* X_{11}^\dagger X_{12} = \Gamma\Sigma^*(\Sigma\Sigma^*)^\dagger \Sigma\Gamma^* = \Gamma\Gamma^*$, the last inequality in (A1) can be rewritten as $X_{22} \succcurlyeq \Gamma\Gamma^*$, and hence there exists a matrix Ω such that $X_{22} = \Gamma\Gamma^* + \Omega\Omega^*$. □

Proof of Proposition 17
To start, for all $x \in \operatorname{Ker}\Gamma^*$, we have $x^*\Sigma\Sigma^*x \leqslant x^*\Gamma\Gamma^*x = 0$; but since $x^*\Sigma\Sigma^*x \geqslant 0$, then $\Sigma^*x = 0$ and so $x \in \operatorname{Ker}\Sigma^*$. Hence, $\operatorname{Ker}\Gamma^* \subseteq \operatorname{Ker}\Sigma^*$, which is equivalent to saying that $\operatorname{Im}\Sigma \subseteq \operatorname{Im}\Gamma$. Then, clearly $\Gamma\Gamma^\dagger\Sigma = \Sigma$, where $\Gamma^\dagger$ denotes the Moore–Penrose inverse of Γ. Setting $\Omega = \Gamma^\dagger\Sigma$ and noticing that $\Omega\Omega^* = \Gamma^\dagger\Sigma\Sigma^*\Gamma^{\dagger *} \preccurlyeq \Gamma^\dagger\Gamma\Gamma^*\Gamma^{\dagger *} \preccurlyeq I$ complete the proof. □

Proof of Theorem 22 (*conclusion*)
The proof of Theorem 22 provided in the main text took provisionally 'constructing a 2-solution from a 1-solution' to be feasible. This is proved here.

Given a $(1,q)$-eventually periodic plant, we provide here the procedure for constructing a 2-solution from a 1-solution to the alternative LMIs (18) in the case where $C_{2i} = I$, $D_{21i} = 0$ for all $i = 1, 2, \ldots, q$. To start, consider the 1-solution (Y_f, Y_b, Y_c) to the alternative LMIs (18), where these LMIs simplify in this case to the following:

$$\sum_{i=0}^{q} \operatorname{trace}(H_i Y_{fi}) + \sum_{i=0}^{q} \operatorname{trace}(W_i Y_{bi}) \geqslant 2\sum_{i=1}^{q} \operatorname{trace}(Y_{ci})$$

$$U_{10} Y_{b0} U_{10}^* = 0$$

$$\begin{bmatrix} V_{10} Y_{f0} V_{10}^* + V_{1q} Y_{fq} V_{1q}^* - F_1 Y_{f1} F_1^* & Y_{c1} \\ Y_{c1}^* & -B_{10} Y_{b0} B_{10}^* - B_{1q} Y_{bq} B_{1q}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{11} Y_{f1} V_{11}^* - F_2 Y_{f2} F_2^* & Y_{c2} \\ Y_{c2}^* & -B_{11} Y_{b1} B_{11}^* \end{bmatrix} \preccurlyeq 0$$

$$\vdots$$

$$\begin{bmatrix} V_{1,q-1} Y_{f,q-1} V_{1,q-1}^* - F_q Y_{fq} F_q^* & Y_{cq} \\ Y_{cq}^* & -B_{1,q-1} Y_{b,q-1} B_{1,q-1}^* \end{bmatrix} \preccurlyeq 0$$

The third LMI in the preceding sequence can be equivalently written as

$$\begin{bmatrix} E_2 & Y_{c1}^* \\ Y_{c1} & E_1 \end{bmatrix} \succcurlyeq 0 \tag{A2}$$

where $E_1 = F_1 Y_{f1} F_1^* - V_{10} Y_{f0} V_{10}^* - V_{1q} Y_{fq} V_{1q}^* \succcurlyeq 0$ and $E_2 = B_{10} Y_{b0} B_{10}^* + B_{1q} Y_{bq} B_{1q}^* \succcurlyeq 0$. Let

$$\Theta_1 = (B_{10} Y_{b0} B_{10}^*)^{1/2} \quad \text{and} \quad \Theta_2 = (B_{1q} Y_{bq} B_{1q}^*)^{1/2}$$

then $E_2 = \Theta_1 \Theta_1^* + \Theta_2 \Theta_2^*$. Assume the most general case where Θ_1 and Θ_2 are both non-zero matrices; the special cases require simple proofs that the reader can easily deduce at this point. By the previous assumption, E_2 is non-zero and so there exists a matrix Θ with full column rank such that $E_2 = [\Theta_1 \ \ \Theta_2][\Theta_1 \ \ \Theta_2]^* = \Theta\Theta^*$. Then clearly there exists a matrix Φ appropriately partitioned as $[\Phi_1 \ \ \Phi_2]$ in accordance with the partitioning of the matrix $[\Theta_1 \ \ \Theta_2]$ such that $\Phi\Phi^* = I$ and $\Theta\Phi = [\Theta_1 \ \ \Theta_2]$; hence, $\Theta\Phi_1 = \Theta_1$ and $\Theta\Phi_2 = \Theta_2$.

The inequality $E_1 \succcurlyeq 0$ is conveniently written as $V_{10} Y_{f0} V_{10}^* + V_{1q} Y_{fq} V_{1q}^* \preccurlyeq F_1 Y_{f1} F_1^*$. Then, following the same argument as that in the proof of Theorem 19, there exists a matrix $\Sigma = [\Sigma_1 \ \ \Sigma_2]$ such that

$$\Sigma\Sigma^* \preccurlyeq I, \ \ F_1 Y_{f1}^{1/2} \Sigma_1 = V_{10} Y_{f0}^{1/2}, \quad \text{and} \quad F_1 Y_{f1}^{1/2} \Sigma_2 = V_{1q} Y_{fq}^{1/2}$$

Note that $E_1 = \Omega\Omega^*$, where $\Omega = F_1 Y_{f1}^{1/2}(I - \Sigma\Sigma^*)^{1/2}$. Applying Proposition 16 to inequality (A2), there exist matrices $\bar{\Omega}$ and $\hat{\Omega}$ of appropriate dimensions such that

$$\begin{bmatrix} E_2 & Y_{c1}^* \\ Y_{c1} & E_1 \end{bmatrix} = \begin{bmatrix} \Theta\Theta^* & \Theta\bar{\Omega}^* \\ \bar{\Omega}\Theta^* & \bar{\Omega}\bar{\Omega}^* + \hat{\Omega}\hat{\Omega}^* \end{bmatrix}$$

Clearly, $\bar{\Omega}\bar{\Omega}^* \preccurlyeq \Omega\Omega^*$, and hence, by Proposition 17, there exists a matrix Ψ such that $\Psi\Psi^* \preccurlyeq I$ and $\Omega\Psi = \bar{\Omega}$. Thus, $Y_{c1} = \Omega\Psi\Theta^*$.

Define Δ, Δ_1 and Δ_2 by $\Delta = \Psi\Phi = [\Psi\Phi_1 \ \ \Psi\Phi_2] = [\Delta_1 \ \ \Delta_2]$; that is, $\Delta_1 = \Psi\Phi_1$ and $\Delta_2 = \Psi\Phi_2$. Note that $\Delta\Delta^* = \Psi\Psi^* \preccurlyeq I$. Also, define $\Omega_1 = \Omega\Delta_1$ and $\Omega_2 = \Omega\Delta_2$. The following are valid:

$$\begin{bmatrix} \Theta_1\Theta_1^* & \Theta_1\Omega_1^* \\ \Omega_1\Theta_1^* & \Omega\Omega^* - \Omega_2\Omega_2^* \end{bmatrix} \succcurlyeq 0 \quad \text{and} \quad \begin{bmatrix} \Theta_2\Theta_2^* & \Theta_2\Omega_2^* \\ \Omega_2\Theta_2^* & \Omega_2\Omega_2^* \end{bmatrix} \succcurlyeq 0 \tag{A3}$$

Note that the first inequality in (A3) is valid since $\Omega\Omega^* - \Omega_2\Omega_2^* \succcurlyeq \Omega_1\Omega_1^*$. Going back to the proof of Theorem 19, and setting Γ in the said proof equal to Δ_2, we define the following matrices:

$$Y_{f1,a} = Y_{f1}^{1/2}(\Sigma_1\Sigma_1^* + (I - \Sigma\Sigma^*)^{1/2}(I - \Delta_2\Delta_2^*)(I - \Sigma\Sigma^*)^{1/2})Y_{f1}^{1/2} \succcurlyeq 0$$

$$Y_{f1,b} = Y_{f1}^{1/2}(\Sigma_2\Sigma_2^* + (I - \Sigma\Sigma^*)^{1/2}\Delta_2\Delta_2^*(I - \Sigma\Sigma^*)^{1/2})Y_{f1}^{1/2} \succcurlyeq 0$$

where $Y_{f1,a} + Y_{f1,b} = Y_{f1}$. Then, we have $\Omega\Omega^* - \Omega_2\Omega_2^* = F_1 Y_{f1,a} F_1^* - V_{10} Y_{f0} V_{10}^*$ and $\Omega_2\Omega_2^* = F_1 Y_{f1,b} F_1^* - V_{1q} Y_{fq} V_{1q}^*$. Thus, inequalities (A3) can be equivalently written as

$$\begin{bmatrix} V_{10} Y_{f0} V_{10}^* - F_1 Y_{f1,a} F_1^* & Y_{c1,a} \\ Y_{c1,a}^* & -B_{10} Y_{b0} B_{10}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{1q} Y_{fq} V_{1q}^* - F_1 Y_{f1,b} F_1^* & Y_{c1,b} \\ Y_{c1,b}^* & -B_{1q} Y_{bq} B_{1q}^* \end{bmatrix} \preccurlyeq 0$$

where $Y_{c1,a} = \Omega_1 \Theta_1^*$ and $Y_{c1,b} = \Omega_2 \Theta_2^*$. Notice that

$$Y_{c1,a} + Y_{c1,b} = [\Omega_1 \;\; \Omega_2][\Theta_1 \;\; \Theta_2]^* = \Omega\Delta\Phi^*\Theta^* = Y_{c1}$$

since $\Delta = \Psi\Phi$ and $\Phi\Phi^* = I$. Consequently, from the preceding, it is obvious that the following LMIs hold:

$$\begin{aligned} &\mathrm{trace}(H_0 Y_{f0}) + \mathrm{trace}(H_1 Y_{f1,a}) + \textstyle\sum_{i=2}^{q} \mathrm{trace}(H_i Y_{fi}) + \mathrm{trace}(H_1 Y_{f1,b}) + \textstyle\sum_{i=0}^{q} \mathrm{trace}(W_i Y_{bi}) \\ &\quad + \mathrm{trace}((W_1)(0)) \geqslant 2 \left(\mathrm{trace}(Y_{c1,a}) + \textstyle\sum_{i=2}^{q} \mathrm{trace}(Y_{ci}) + \mathrm{trace}(Y_{c1,b})\right) \end{aligned}$$

$$\begin{bmatrix} V_{10} Y_{f0} V_{10}^* - F_1 Y_{f1,a} F_1^* & Y_{c1,a} \\ Y_{c1,a}^* & -B_{10} Y_{b0} B_{10}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{11} Y_{f1,a} V_{11}^* + V_{11} Y_{f1,b} V_{11}^* - F_2 Y_{f2} F_2^* & Y_{c2} \\ Y_{c2}^* & -B_{11} Y_{b1} B_{11}^* - B_{11}(0) B_{11}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{12} Y_{f2} V_{12}^* - F_3 Y_{f3} F_3^* & Y_{c3} \\ Y_{c3}^* & -B_{12} Y_{b2} B_{12}^* \end{bmatrix} \preccurlyeq 0$$

$$\vdots$$

$$\begin{bmatrix} V_{1,q-1} Y_{f,q-1} V_{1,q-1}^* - F_q Y_{fq} F_q^* & Y_{cq} \\ Y_{cq}^* & -B_{1,q-1} Y_{b,q-1} B_{1,q-1}^* \end{bmatrix} \preccurlyeq 0$$

$$\begin{bmatrix} V_{1q} Y_{fq} V_{1q}^* - F_1 Y_{f1,b} F_1^* & Y_{c1,b} \\ Y_{c1,b}^* & -B_{1q} Y_{bq} B_{1q}^* \end{bmatrix} \preccurlyeq 0$$

Thus, from the given 1-solution, we have constructed a 2-solution to the alternative LMIs (18), namely the triplet $(\bar{Y}_f, \bar{Y}_b, \bar{Y}_c)$, where $\bar{Y}_f = \mathrm{diag}(Y_{f0}, Y_{f1,a}, Y_{f2}, \ldots, Y_{fq}, Y_{f1,b})$, $\bar{Y}_b = \mathrm{diag}(Y_{b0}, Y_{b1}, \ldots, Y_{bq}, 0)$, and $\bar{Y}_c = \mathrm{diag}(0, Y_{c1,a}, Y_{c2}, \ldots, Y_{cq}, Y_{c1,b})$. □

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