Model Reduction of Nonstationary LPV Systems

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***Abstract*—This paper focuses on the model reduction of nonstationary linear parameter-varying (NSLPV) systems. We provide a generalization of the balanced truncation procedure for the model reduction of stable NSLPV systems, along with a priori error bounds. Then, for illustration purposes, this method is applied to reduce the model of a two-mass translational system. Furthermore, we give an approach for the model reduction of stabilizable and detectable systems, which requires the development and use of coprime factorizations for NSLPV models. For the general class of eventually periodic LPV systems, which includes periodic and finite horizon systems as special cases, our results can be explicitly computed using semidefinite programming.**

***Index Terms*—Balanced truncation, coprime factors reduction, linear parameter-varying (LPV) systems, model reduction, time-varying systems.**

I. Introduction

THIS paper deals with the model reduction of nonstationary linear parameter-varying (NSLPV) systems. Our interest in LPV models is motivated by the desire to control nonlinear systems along prespecified trajectories. LPV models arise naturally in such scenarios as a method to capture the possible nonlinear dynamics, while maintaining a model that is amenable to control synthesis. Frequently, when pursuing such an LPV formulation, one ends up with models of relatively large dimension. Accordingly, finding control syntheses for such models, which usually involves solving a number of linear operator inequalities as discussed in [1], requires substantial computation. For this reason, developing a theory that provides systematic methods of approximating such models is beneficial. Our methods may also be of potential use in situations (for instance dynamical networks) where the topological structure is to be preserved.

Specifically, the types of plant models we consider in this paper are of the form

$$\begin{aligned} x(k+1) &= A(\delta(k),k)x(k) + B(\delta(k),k)w(k) \\ z(k) &= C(\delta(k),k)x(k) + D(\delta(k),k)w(k) \end{aligned}$$

where $A(\cdot,\cdot), B(\cdot,\cdot), C(\cdot,\cdot)$, and $D(\cdot,\cdot)$ are matrix-valued functions that are known *a priori*. The variable k is time, and $\delta(k) := (\delta_1(k),\ldots,\delta_d(k))$ is a vector of real scalar parameters. In this paper, we are concerned only with the subclass of NSLPV models in which the dependence of the matrix functions A, B, C, and D on the parameters δ_i is rational and given in terms of a feedback coupling. Such models are commonly referred to as LFT systems and are basically the straightforward generalization of the LPV systems first introduced in [2], [3]. These models can be interpreted in at least two interesting ways. First, as stated before, NSLPV models arise naturally in control problems of nonlinear systems along prespecified trajectories; in such a case, the parameters δ_i are not uncertain but rather available for measurement at each time k. Alternatively, such models can be regarded as linear time-varying (LTV) plants subject to parametric time-varying uncertainty $\delta(k)$; these have uses in LTV robust control problems.

The paper focuses on the model reduction of stable as well as stabilizable and detectable NSLPV models. In the case of stable models, we utilize the theory of generalized gramians to define the notion of balanced realizations for NSLPV systems. We also examine the balanced truncation method in detail and derive error bounds for such a reduction process. These error bounds are generalizations of their LTV counterparts given in [4], [5], and even in the standard time-varying case, the main results here can give tighter error bounds than what is currently available. We demonstrate the applicability of these results in an example of a two-mass translational system. As for stabilizable and detectable models, the approach used is motivated by the work in [6], [7], where a coprime factorization method for reducing generalized state-space systems containing stationary LPV and uncertain systems is proposed; the method itself is a generalization of the work in [8] for standard systems. Also, we specialize these results to the subclass of eventually periodic LPV models introduced in [1]; these are aperiodic for an initial amount of time and contain both finite horizon and periodic systems as special cases. The contributions of the paper are as follows.

- Generalization of the balanced truncation model reduction procedure as well as the coprime factors reduction method to the class of NSLPV systems.
- Several results on the worst-case balanced truncation error. These results when restricted to the purely time-varying case (i.e., no parameters) provide the least conservative error bounds currently available in the literature.
- Operator theoretic machinery is developed in the context of standard robust control tools for working with NSLPV models.

This paper deploys a combination of recent work on NSLPV models in [1] and new work on model reduction using balanced truncation for standard LTV systems in [4], [5]. The approach is motivated by the work in [9] on the generalization of balanced truncation to stationary multidimensional systems, and that in [10] on discrete time model reduction of standard LTI systems. The basic approach behind balanced truncation originates in [11], and the by-now famous error bounds associated with this method in the LTI case were first demonstrated in [12], [13].

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The paper is organized as follows: In Section II, we establish notation and collect some needed definitions; in Section III, we introduce NSLPV models and discuss well-posedness and two forms of stability; in Section IV, we provide the balanced truncation procedure and prove the error bound results of the paper; in Section V, we apply the balanced truncation results to a two-mass translational system exhibiting eventually periodic dynamics; in Section VI, we generalize the coprime factors reduction method to the class of NSLPV systems; we then conclude with a summary statement.

II. Preliminaries

The set of real $n \times m$ matrices and that of real symmetric $n \times n$ matrices are denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ respectively. The maximum singular value of a matrix X is denoted by $\bar{\sigma}(X)$.

The main Hilbert space of interest in this paper is formed from an infinite sequence of Euclidean spaces $(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots)$, and is denoted by $\ell_2(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \ldots)$, or just $\ell_2(\mathbb{R}^n)$ for short. It is defined as the subspace of the Hilbert space direct sum $\oplus_{k=0}^{\infty}\mathbb{R}^{n(k)}$ consisting of elements $x = (x(0), x(1), x(2), \ldots)$, with each $x(k) \in \mathbb{R}^{n(k)}$, such that $\|x\|^2 = \sum_{k=0}^{\infty} x(k)^* x(k) < \infty$. The inner product of x, y in $\ell_2(\mathbb{R}^n)$ is hence defined as the sum $\langle x, y\rangle = \sum_{k=0}^{\infty} x(k)^* y(k)$. Another Hilbert space of interest here is the direct sum of the aforementioned Hilbert spaces, namely $\ell_2(\mathbb{R}^{n_1}) \oplus \ell_2(\mathbb{R}^{n_2}) \oplus \cdots \oplus \ell_2(\mathbb{R}^{n_p})$, for some $p > 1$. The linear space $\oplus_{k=0}^{\infty}\mathbb{R}^{n(k)}$ will be denoted by $\ell(\mathbb{R}^n)$, and another linear space used is $\ell(\mathbb{R}^{n_1}) \oplus \ell(\mathbb{R}^{n_2}) \oplus \cdots \oplus \ell(\mathbb{R}^{n_p})$. When the spatial structure and dimensions are either evident or irrelevant to the discussion, we will often abbreviate these denotations simply by ℓ_2 and ℓ.

Given E and F, which represent any two of the previously defined Hilbert spaces, we denote the space of bounded linear operators mapping E to F by $\mathcal{L}(E, F)$, and that of bounded linear causal operators by $\mathcal{L}_c(E, F)$. We shorten these notations to $\mathcal{L}(E)$ and $\mathcal{L}_c(E)$ when E equals F. When the spatial structures and dimensions are not pertinent to the discussion, we simply use the notations $\mathcal{L}(\ell_2)$ and $\mathcal{L}_c(\ell_2)$. If X is in $\mathcal{L}(E, F)$, we denote the E to F induced norm of X by $\|X\|_{E\to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. When an operator $X \in \mathcal{L}(E)$ is self-adjoint, we use $X \prec 0$ to mean it is negative definite; that is there exists a number $\alpha > 0$ such that, for all nonzero $x \in E$, the inequality $\langle x, Xx\rangle < -\alpha\|x\|^2$ holds. Given any two of the previous linear spaces, which we call $\hat{E}$ and $\hat{F}$ for simplicity, we define the algebra $\mathcal{L}_e(\hat{E}, \hat{F})$ to be the space of linear causal operators mapping $\hat{E}$ to $\hat{F}$, and equipped with the pointwise topology with respect to the standard matrix representation. Similar abbreviations as before apply to this space.

A key operator used in the paper is the unilateral shift Z, defined as follows:

$$Z:\quad \ell_2\left(\mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots\right) \to \ell_2\left(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots\right)$$
$$(a(1), a(2), \ldots) \overset{Z}{\longmapsto} (0, a(1), a(2), \ldots).$$

Clearly, this definition is extendable to ℓ, and in the sequel, we will not distinguish between these mappings. If S_i is a sequence of operators, then $\mathrm{diag}(S_i)$ denotes their block-diagonal augmentation. Given a time-varying dimension $n(k)$, we define the notation $I_{\ell_2}^n := \mathrm{diag}(I_{n(0)}, I_{n(1)}, I_{n(2)}, \ldots)$, where $I_{n(k)}$ is an $n(k) \times n(k)$ identity matrix.

Following the notation and approach in [14], we make the following definitions. First, we say a linear operator Q mapping $\ell(\mathbb{R}^{m(0)}, \mathbb{R}^{m(1)}, \ldots)$ to $\ell(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \ldots)$ is *block-diagonal* if there exists a sequence of matrices $Q(k)$ in $\mathbb{R}^{n(k)\times m(k)}$ such that, for all w, z, if $z = Qw$, then $z(k) = Q(k)w(k)$. Then, Q has the representation $\mathrm{diag}(Q(0), Q(1), Q(2), \ldots)$. A *diagonal* operator is a block-diagonal operator where each of the matrix blocks is diagonal.

Suppose F, G, R, and S are block-diagonal operators, and let A be a *partitioned* operator of the form

$$A = \begin{bmatrix} F & G \\ R & S \end{bmatrix}.$$

Then, we define the following notation:

$$[\![A]\!] := \mathrm{diag}\left(\begin{bmatrix} F(0) & G(0) \\ R(0) & S(0) \end{bmatrix}, \begin{bmatrix} F(1) & G(1) \\ R(1) & S(1) \end{bmatrix}, \ldots\right)$$

which we call the *diagonal realization* of A. Clearly, for any given operator A of this particular structure, [[A]] is simply A with the rows and columns permuted appropriately so that

$$[\![A]\!]_k = \begin{bmatrix} F(k) & G(k) \\ R(k) & S(k) \end{bmatrix}.$$

From this definition, it is easy to see that $[\![A+B]\!] = [\![A]\!] + [\![B]\!]$ and $[\![AC]\!] = [\![A]\!][\![C]\!]$ hold for appropriately dimensioned operators, and similarly that $A \prec \beta I$ holds if and only if $[\![A]\!] \prec \beta I$, where β is a scalar. Namely, the $[\![\bullet]\!]$ operation is a homomorphism from partitioned operators with block-diagonal entries to block-diagonal operators. Last, we will find the following formal notation very useful:

$$\Lambda \star P := P_{21}(I - \Lambda P_{11})^{-1}\Lambda P_{12} + P_{22}$$

where

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right].$$

III. NSLPV Systems

We now review NSLPV models. The reader is referred to [1] for further treatment of the theory. Let G be a linear time-varying discrete-time system defined by the following state–space equation:

$$\begin{bmatrix} x(k+1) \\ \alpha(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{ss}(k) & A_{sp}(k) & B_s(k) \\ A_{ps}(k) & A_{pp}(k) & B_p(k) \\ C_s(k) & C_p(k) & D(k) \end{bmatrix} \begin{bmatrix} x(k) \\ \beta(k) \\ w(k) \end{bmatrix} \tag{1}$$

where $x(0) = 0$ and $w \in \ell_2$. The vector-valued signals $x(k), \alpha(k), \beta(k), z(k)$, and $w(k)$ are real and have time-varying dimensions, with the constraint that $\dim(\beta(k)) = \dim(\alpha(k))$. We denote the dimensions of these signals by $n_0(k), n(k), n(k), n_z(k)$, and $n_w(k)$, respectively. We assume that all the state space matrices are uniformly bounded functions

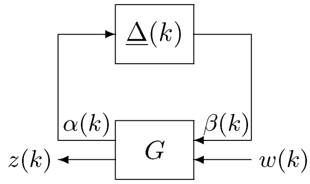


Fig. 1. Interconnection of G with $\underline{\Delta}(k)$.

of time. Given a scalar sequence $\delta_1(k), \ldots, \delta_d(k)$ and associated dimensions $n_1(k), \ldots, n_d(k)$ satisfying $\sum_{i=1}^{d} n_i(k) = n(k)$, we define the diagonal matrix $\underline{\Delta}(k)$ as

$$\underline{\Delta}(k) := \operatorname{diag}(\delta_1(k) I_{n_1(k)}, \ldots, \delta_d(k) I_{n_d(k)}) \in \mathbb{R}^{n(k)\times n(k)}.$$

Also, we constrain $\bar{\sigma}(\underline{\Delta}(k)) \le 1$ for all $k \ge 0$. We will be concerned with the arrangement in Fig. 1, where G and $\underline{\Delta}(k)$ are connected in feedback. This system can be expressed formally by

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = H(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{2}$$

where

$$H(k) = \begin{bmatrix} A_{ss}(k) & B_s(k) \\ C_s(k) & D(k) \end{bmatrix} + \begin{bmatrix} A_{sp}(k) \\ C_p(k) \end{bmatrix} \underline{\Delta}(k) \times (I - A_{pp}(k)\underline{\Delta}(k))^{-1} [A_{ps}(k) \quad B_p(k)]. \tag{3}$$

We assume $A_{pp}(k)$ such that $I - A_{pp}(k)\underline{\Delta}(k)$ is invertible for all $k \ge 0$ so that the LFT in (3) is well-defined at each time k. This well-posedness condition guarantees that there are unique solutions in ℓ to (2). We will refer to the mapping $w \mapsto z$ in (2) as the system G_δ. Hence, G_δ is a linear time-varying system with rational state-space parameter dependence formulated in an LFT framework, where the time-varying parameters δ_i act on the system G through the linear fractional feedback channels (α, β).

Using the previously defined notation, clearly the matrix sequences $A_{ss}(k), B_s(k), C_s(k)$, and $D(k)$ from (1) define bounded block-diagonal operators. The blocks of the matrix $\underline{\Delta}(k)$ naturally partition $\alpha(k)$ and $\beta(k)$ into d separate vector-valued channels, conformably with which we partition the following state–space matrices, such that:

$$A_{sp}(k) = [\, A_{sp}^1(k) \quad A_{sp}^2(k) \quad \cdots \quad A_{sp}^d(k) \,]$$
$$A_{pp}(k) = \begin{bmatrix} A_{pp}^{11}(k) & \cdots & A_{pp}^{1d}(k) \\ \vdots & \ddots & \vdots \\ A_{pp}^{d1}(k) & \cdots & A_{pp}^{dd}(k) \end{bmatrix}$$
$$A_{ps}(k) = \begin{bmatrix} A_{ps}^1(k) \\ A_{ps}^2(k) \\ \vdots \\ A_{ps}^d(k) \end{bmatrix} \qquad B_p(k) = \begin{bmatrix} B_p^1(k) \\ B_p^2(k) \\ \vdots \\ B_p^d(k) \end{bmatrix}$$
$$C_p(k) = [\, C_p^1(k) \quad C_p^2(k) \quad \cdots \quad C_p^d(k) \,] \tag{4}$$

where $A_{sp}^i(k) \in \mathbb{R}^{n_0(k+1)\times n_i(k)}, A_{pp}^{ij}(k) \in \mathbb{R}^{n_i(k)\times n_j(k)}$, $A_{ps}^i(k) \in \mathbb{R}^{n_i(k)\times n_0(k)}$, $B_p^i(k) \in \mathbb{R}^{n_i(k)\times n_w(k)}$, and $C_p^i(k) \in \mathbb{R}^{n_z(k)\times n_i(k)}$. The matrix sequence of each of the elements of the state space matrices in (4) defines a bounded block-diagonal operator; and so we construct from the sequence of each of these state space matrices a partitioned operator, each of whose elements is block-diagonal and defined in the obvious way. For instance, the matrix sequences $A_{sp}^1(k), \ldots, A_{sp}^d(k)$ define block-diagonal operators that compose the partitioned operator A_{sp}. With Z being the shift, we rewrite our system equations as

$$\begin{bmatrix} x \\ \alpha \\ z \end{bmatrix} = \begin{bmatrix} ZA_{ss} & ZA_{sp} & ZB_s \\ A_{ps} & A_{pp} & B_p \\ C_s & C_p & D \end{bmatrix} \begin{bmatrix} x \\ \beta \\ w \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \operatorname{diag}\left(I_{\ell_2}^{n_0}, \Delta_1, \ldots, \Delta_d\right) \begin{bmatrix} x \\ \alpha \end{bmatrix} = \Delta \begin{bmatrix} x \\ \alpha \end{bmatrix} \tag{6}$$

where $x \in \ell(\mathbb{R}^{n_0}), w \in \ell_2(\mathbb{R}^{n_w}), z \in \ell(\mathbb{R}^{n_z}), \beta = (\beta_1, \ldots, \beta_d), \alpha = (\alpha_1, \ldots, \alpha_d), \beta_i, \alpha_i \in \ell(\mathbb{R}^{n_i})$, and

$$\Delta_i = \operatorname{diag}(\delta_i(0) I_{n_i(0)}, \delta_i(1) I_{n_i(1)}, \delta_i(2) I_{n_i(2)}, \ldots).$$

We now introduce some convenient definitions and notations that will be used extensively in the sequel. To start, we define

$$\tilde{Z} := \begin{bmatrix} Z & \\ & I \end{bmatrix} \quad A := \begin{bmatrix} A_{ss} & A_{sp} \\ A_{ps} & A_{pp} \end{bmatrix}$$
$$B := \begin{bmatrix} B_s \\ B_p \end{bmatrix} \quad C := [C_s \quad C_p]$$

where $\tilde{Z}$ is partitioned similarly to operator A. Note that this partitioning is also conformable to that of $\Delta = \operatorname{diag}(\Delta_s, \Delta_p)$, where

$$\Delta_s = I_{\ell_2}^{n_0} \quad \text{and} \quad \Delta_p = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_d).$$

Moreover, we define $\ell_2^{(n_0,\ldots,n_d)} := \ell_2(\mathbb{R}^{n_0}) \oplus \ell_2(\mathbb{R}^{n_1}) \oplus \cdots \oplus \ell_2(\mathbb{R}^{n_d})$ and

$$\mathbf{\Delta} := \left\{ \Delta \in \mathcal{L}_c\left(\ell_2^{(n_0,\ldots,n_d)}\right) : \Delta \text{ is partitioned as in (6) and } \|\Delta\| \le 1 \right\}.$$

We formally define the system G_δ by $G_\delta = \Delta \star G$, where $\Delta \in \mathbf{\Delta}$ and

$$G = \left[\begin{array}{c|c} \tilde{Z}A & \tilde{Z}B \\ \hline C & D \end{array}\right].$$

When the relevant inverse exists, then G_δ is well-posed, in which case we can rewrite (5) and (6) in the form $z = G_\delta w$, and hence G_δ can be viewed as a mapping in $\mathcal{L}_e(\ell)$. The set of systems G_δ for all $\Delta \in \mathbf{\Delta}$ defines an NSLPV model $\mathcal{G}_\delta$, which we formally express as

$$\mathcal{G}_\delta = \mathbf{\Delta} \star G = \{\Delta \star G : \Delta \in \mathbf{\Delta}\}.$$

The system realization of NSLPV mode $\mathcal{G}_\delta$ will be denoted by $(A, B, C, D; \mathbf{\Delta})$.

A. Well-Posedness

We now define a basic notion of well-posedness for NSLPV models.

Definition 1: An NSLPV model is well-posed if $I - \Delta\tilde{Z}A$ is invertible in $\mathcal{L}_e(\ell)$ for all $\Delta \in \mathbf{\Delta}$.

Proposition 2: The following are true.

i) Given a block-diagonal operator S in $\mathcal{L}_e(\ell)$, then $I - ZS$ is invertible in $\mathcal{L}_e(\ell)$.

ii) For all $\Delta = \mathrm{diag}(I_{\ell_2}^{n_0}, \Delta_p) \in \mathbf{\Delta}, I - \Delta\tilde{Z}A$ is invertible in $\mathcal{L}_e(\ell)$ if and only if $I - \Delta_p A_{pp}$ is invertible in $\mathcal{L}_e(\ell)$.

The following proof of this proposition is inspired by those of similar results in [7].

Proof: To start, it is not difficult to verify that the infinite series $\sum_{i=0}^{\infty}(ZS)^i$ defines a unique element in $\mathcal{L}_e(\ell)$, which is indeed the inverse of $I - ZS$; this proves part i) of the claim.

Regarding the "if" direction of part ii), since $I - \Delta_p A_{pp}$ is invertible in $\mathcal{L}_e(\ell)$, we can factorize $I - \Delta\tilde{Z}A$ as follows:

$$I - \Delta\tilde{Z}A = \begin{bmatrix} I - ZA_{ss} & -ZA_{sp} \\ -\Delta_p A_{ps} & I - \Delta_p A_{pp} \end{bmatrix} = \begin{bmatrix} I & -ZA_{sp}(I - \Delta_p A_{pp})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - ZR & 0 \\ -\Delta_p A_{ps} & I - \Delta_p A_{pp} \end{bmatrix}$$

where $R = A_{ss} + A_{sp}(I - \Delta_p A_{pp})^{-1}\Delta_p A_{ps}$ is block-diagonal in $\mathcal{L}_e(\ell)$. As $I - ZR$ is invertible in $\mathcal{L}_e(\ell)$ by part i), then clearly from the aforementioned factorization $I - \Delta\tilde{Z}A$ is invertible as well. To prove the "only if" direction, notice that since, by part i), $I - ZA_{ss}$ is invertible in $\mathcal{L}_e(\ell)$, we can write the following factorization:

$$I - \Delta\tilde{Z}A = \begin{bmatrix} I & 0 \\ -\Delta_p A_{ps}(I - ZA_{ss})^{-1} & I \end{bmatrix} \begin{bmatrix} I - ZA_{ss} & -ZA_{sp} \\ 0 & H \end{bmatrix}$$

where $H = I - \Delta_p A_{pp} - \Delta_p A_{ps}(I - ZA_{ss})^{-1}ZA_{sp}$, or equivalently

$$H = I \star \left[\begin{array}{c|c} ZA_{ss} & ZA_{sp} \\ \hline -\Delta_p A_{ps} & I - \Delta_p A_{pp} \end{array}\right].$$

Clearly, from the last factorization, the invertibility of $I - \Delta\tilde{Z}A$ implies that H is invertible in $\mathcal{L}_e(\ell)$, which in turn implies that $I - \Delta_p A_{pp}$ is invertible. ∎

The next result stems from Definition 1 and Proposition 2.

Corollary 3: An NSLPV model is well-posed if and only if operator $I - \Delta_p A_{pp}$ is invertible in $\mathcal{L}_e(\ell)$ for all $\Delta = \mathrm{diag}(I_{\ell_2}^{n_0}, \Delta_p) \in \mathbf{\Delta}$.

Note that this result is deducible from (3) as noted before. Thus, an NSLPV model $\mathcal{G}_\delta$ is well-posed if, for each $\Delta \in \mathbf{\Delta}$, the corresponding system G_δ is well-posed, i.e., $I - \Delta_p A_{pp}$ is invertible in $\mathcal{L}_e(\ell)$.

B. Stability

This section tackles the concepts of stability that are essential to our work.

Definition 4: An NSLPV model is ℓ_2-stable if $I - \Delta\tilde{Z}A$ is invertible in $\mathcal{L}_c(\ell_2)$ for all $\Delta \in \mathbf{\Delta}$.

Consequently, an ℓ_2-stable NSLPV model constitutes a set of linear bounded causal mappings from $\ell_2(\mathbb{R}^{n_w})$ to $\ell_2(\mathbb{R}^{n_z})$, namely $G_\delta = w \mapsto z \in \mathcal{L}_c(\ell_2(\mathbb{R}^{n_w}), \ell_2(\mathbb{R}^{n_z}))$ for all $\Delta \in \mathbf{\Delta}$.

At this point, we define $\mathcal{T}$ as the set of operators $T \in \mathcal{L}_c(\ell_2^{(n_0,\ldots,n_d)})$ that have bounded inverses and are of the form $T = \mathrm{diag}(T_0, T_1, \ldots, T_d)$, where each $T_i \in \mathcal{L}_c(\ell_2(\mathbb{R}^{n_i}))$ is block-diagonal so that $[\![T_i]\!]_k = T_i(k) \in \mathbb{R}^{n_i(k)\times n_i(k)}$. Observe that $\mathcal{T}$ is a commutant of $\mathbf{\Delta}$. Moreover, we define the subset $\mathcal{X}$ of $\mathcal{T}$ by $\mathcal{X} = \{X \succ 0 : X \in \mathcal{T}\}$.

Definition 5: An NSLPV model is strongly ℓ_2-stable if there exists $P \in \mathcal{X}$ satisfying

$$APA^* - \tilde{Z}^* P\tilde{Z} \prec 0. \tag{7}$$

The following lemma asserts that strongly ℓ_2-stable NSLPV models constitute a subset of the ℓ_2-stable ones.

Lemma 6: A strongly ℓ_2-stable NSLPV model is also ℓ_2-stable; however, the converse is not true in general.

The proof parallels the standard case and is hence omitted.

Remark 7: We know that, under ℓ_2-stability, the norm $\|(I - \Delta\tilde{Z}A)^{-1}\|$ is bounded for all $\Delta \in \mathbf{\Delta}$. However, this boundedness is *not* necessarily uniform. On the other hand, strong ℓ_2-stability guarantees that this norm is uniformly bounded; this is clearly shown by the following norm condition which is easily derived from (7):

$$\|(I - \Delta\tilde{Z}A)^{-1}\| \leq \frac{\left\|P^{-\frac{1}{2}}\right\| \cdot \left\|P^{\frac{1}{2}}\right\|}{1 - \left\|P^{-\frac{1}{2}}\tilde{Z}AP^{\frac{1}{2}}\right\|} < \infty, \text{ for all } \Delta \in \mathbf{\Delta}$$

where P is any solution in $\mathcal{X}$ to inequality (7).

One of the key features of strongly ℓ_2-stable NSLPV models is that they can always be represented by an equivalent *balanced* realization, as we will show next. But first, we need to define the balanced realizations of an NSLPV model.

Definition 8: An NSLPV system realization is ***balanced*** if there exists a ***diagonal*** operator $\Sigma \in \mathcal{X}$ satisfying

$$A\Sigma A^* - \tilde{Z}^*\Sigma\tilde{Z} + BB^* \prec 0 \tag{8}$$

$$A^*\tilde{Z}^*\Sigma\tilde{Z}A - \Sigma + C^*C \prec 0. \tag{9}$$

Lemma 9: An NSLPV model can be equivalently represented by a balanced realization if and only if it is strongly ℓ_2-stable.

Proof: Consider a strongly ℓ_2-stable NSLPV model $(A, B, C, D; \mathbf{\Delta})$. This is equivalent to the existence of $P \in \mathcal{X}$ satisfying (7), which in turn is equivalent to the existence of operators $X, Y \in \mathcal{X}$ solving the generalized Lyapunov inequalities

$$AXA^* - \tilde{Z}^*X\tilde{Z} + BB^* \prec 0 \quad A^*\tilde{Z}^*Y\tilde{Z}A - Y + C^*C \prec 0.$$

Clearly, these two conditions are themselves equivalent. Now we define the operator $T \in \mathcal{T}$ by

$$T := \Sigma^{\frac{1}{2}}U^*X^{-\frac{1}{2}}$$

where unitary operator $U \in \mathcal{T}$ and diagonal operator $\Sigma \in \mathcal{X}$ are obtained by performing a singular value decomposition on $X^{\frac{1}{2}}YX^{\frac{1}{2}}$, namely $U\Sigma^2U^* = X^{1/2}YX^{1/2}$. Then, the following holds:

$$TXT^* = (T^*)^{-1}YT^{-1} = \Sigma.$$

As a result, the equivalent realization

$$((\tilde{Z}^*T\tilde{Z})AT^{-1}, (\tilde{Z}^*T\tilde{Z})B, CT^{-1}, D; \mathbf{\Delta})$$

is obviously balanced.

IV. Model Reduction of Strongly ℓ_2-Stable NSLPV Systems

This section focuses on the balanced truncation model reduction of strongly ℓ_2-stable NSLPV systems. It is divided into three subsections: The first presents a precise formulation of the balanced truncation problem; the second gives upper bounds on the error induced in such a reduction process; and the last deals with eventually periodic LPV systems and delivers guaranteed finite error bounds for the balanced truncation of such systems.

A. Balanced Truncation

Consider a strongly ℓ_2-stable NSLPV system $\mathcal{G}_\delta$ with balanced realization $(A, B, C, D; \mathbf{\Delta})$ and generalized diagonal gramian $\Sigma \in \mathcal{X}$ satisfying both of the generalized Lyapunov inequalities (8) and (9). Recall that $\Sigma = \text{diag}(\Sigma_0, \Sigma_1, \ldots, \Sigma_d)$, where each $\Sigma_i = \text{diag}(\Sigma_i(0), \Sigma_i(1), \Sigma_i(2), \ldots)$, and $\Sigma_i(k)$ is a diagonal positive definite matrix in $\mathbb{S}^{n_i(k)}$. We assume without loss of generality that, in each block $\Sigma_i(k)$, the diagonal entries are ordered with the largest first. Now given the integers $r_i(k)$ such that $0 \le r_i(k) \le n_i(k)$ for all $k \ge 0$, we partition each of the matrices $\Sigma_i(k)$ into two sub-blocks $\Gamma_i(k) \in \mathbb{S}^{r_i(k)}$ and $\Omega_i(k) \in \mathbb{S}^{n_i(k)-r_i(k)}$ so that

$$\Sigma_i = \begin{bmatrix}\!\begin{bmatrix} \Gamma_i & 0 \\ 0 & \Omega_i \end{bmatrix}\!\end{bmatrix} \tag{10}$$

where Γ_i and Ω_i are block-diagonal operators. Note that, since $r_i(k)$ is allowed to be equal to zero or $n_i(k)$ at any time k, it is possible to have one of the matrices $\Omega_i(k)$ or $\Gamma_i(k)$ with zero dimension; this corresponds to the case where either zero states or all states are truncated at a particular k. Allowing for matrices with no entries, although a slight abuse of notation, will be very helpful in the manipulations of the sequel. We define the operators Γ and Ω to have a similar structure to that of Σ, namely $\Gamma = \text{diag}(\Gamma_0, \Gamma_1, \ldots, \Gamma_d)$ and $\Omega = \text{diag}(\Omega_0, \Omega_1, \ldots, \Omega_d)$. The singular values corresponding to the states and parameters that will be truncated are in Ω.

At this point, we want to partition A, B, and C conformably with the partitioning of Σ. Recall from Section III that A, B, and C have the following forms:

$$A = \begin{bmatrix} A_{ss} & A_{sp}^1 & \cdots & A_{sp}^d \\ A_{ps}^1 & A_{pp}^{11} & \cdots & A_{pp}^{1d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{ps}^d & A_{pp}^{d1} & \cdots & A_{pp}^{dd} \end{bmatrix} \quad B = \begin{bmatrix} B_s \\ B_p^1 \\ \vdots \\ B_p^d \end{bmatrix}$$
$$C = [\, C_s \quad C_p^1 \quad \cdots \quad C_p^d \,]$$

where each of the elements of these partitioned system operators is block-diagonal. Note further that

$$[\![A]\!]_k = \begin{bmatrix} A_{ss}(k) & A_{sp}^1(k) & \cdots & A_{sp}^d(k) \\ A_{ps}^1(k) & A_{pp}^{11}(k) & \cdots & A_{pp}^{1d}(k) \\ \vdots & \vdots & \ddots & \vdots \\ A_{ps}^d(k) & A_{pp}^{d1}(k) & \cdots & A_{pp}^{dd}(k) \end{bmatrix}$$
$$[\![B]\!]_k = [\, B_s^*(k) \quad B_p^{1^*}(k) \quad \cdots \quad B_p^{d^*}(k) \,]^*$$
$$[\![C]\!]_k = [\, C_s(k) \quad C_p^1(k) \quad \cdots \quad C_p^d(k) \,]. \tag{11}$$

Let us now focus on the matrices $A_{ss}(k), B_s(k), C_s(k)$ and partition them in accordance with the partitioning of $\Sigma_0(k) = \text{diag}(\Gamma_0(k), \Omega_0(k))$ and $\Sigma_0(k+1) = \text{diag}(\Gamma_0(k+1), \Omega_0(k+1))$ so that $C_s(k) = [\hat{C}_s(k) \;\; C_{s_2}(k)]$

$$A_{ss}(k) = \begin{bmatrix} \hat{A}_{ss}(k) & A_{ss_{12}}(k) \\ A_{ss_{21}}(k) & A_{ss_{22}}(k) \end{bmatrix} \quad B_s(k) = \begin{bmatrix} \hat{B}_s(k) \\ B_{s_2}(k) \end{bmatrix}$$

where $\hat{A}_{ss}(k) \in \mathbb{R}^{r_0(k+1)\times r_0(k)}$, $\hat{B}_s(k) \in \mathbb{R}^{r_0(k+1)\times n_w(k)}$, and $\hat{C}_s(k) \in \mathbb{R}^{n_z(k)\times r_0(k)}$. Hence, we have

$$A_{ss} = \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{ss} & A_{ss_{12}} \\ A_{ss_{21}} & A_{ss_{22}} \end{bmatrix}\!\end{bmatrix} \quad B_s = \begin{bmatrix}\!\begin{bmatrix} \hat{B}_s \\ B_{s_2} \end{bmatrix}\!\end{bmatrix}$$
$$C_s = [\![\hat{C}_s \quad C_{s_2}]\!]$$

where each of the elements is block-diagonal. Similarly, the other elements of the system matrices in (11) are partitioned compatibly with the partitioning of the associated $\Sigma_i(k)$ so that

$$A = \begin{bmatrix} \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{ss} & A_{ss_{12}} \\ A_{ss_{21}} & A_{ss_{22}} \end{bmatrix}\!\end{bmatrix} & \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{sp}^1 & A_{sp_{12}}^1 \\ A_{sp_{21}}^1 & A_{sp_{22}}^1 \end{bmatrix}\!\end{bmatrix} & \cdots & \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{sp}^d & A_{sp_{12}}^d \\ A_{sp_{21}}^d & A_{sp_{22}}^d \end{bmatrix}\!\end{bmatrix} \\ \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{ps}^1 & A_{ps_{12}}^1 \\ A_{ps_{21}}^1 & A_{ps_{22}}^1 \end{bmatrix}\!\end{bmatrix} & \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{pp}^{11} & A_{pp_{12}}^{11} \\ A_{pp_{21}}^{11} & A_{pp_{22}}^{11} \end{bmatrix}\!\end{bmatrix} & \cdots & \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{pp}^{1d} & A_{pp_{12}}^{1d} \\ A_{pp_{21}}^{1d} & A_{pp_{22}}^{1d} \end{bmatrix}\!\end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{ps}^d & A_{ps_{12}}^d \\ A_{ps_{21}}^d & A_{ps_{22}}^d \end{bmatrix}\!\end{bmatrix} & \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{pp}^{d1} & A_{pp_{12}}^{d1} \\ A_{pp_{21}}^{d1} & A_{pp_{22}}^{d1} \end{bmatrix}\!\end{bmatrix} & \cdots & \begin{bmatrix}\!\begin{bmatrix} \hat{A}_{pp}^{dd} & A_{pp_{12}}^{dd} \\ A_{pp_{21}}^{dd} & A_{pp_{22}}^{dd} \end{bmatrix}\!\end{bmatrix} \end{bmatrix}$$
$$B = [\, [\![\hat{B}_s^* \quad B_{s_2}^*]\!] \quad [\![\hat{B}_p^{1^*} \quad B_{p_2}^{1^*}]\!] \quad \cdots \quad [\![\hat{B}_p^{d^*} \quad B_{p_2}^{d^*}]\!] \,]^*$$

$$C = [\, [\![\hat{C}_s \quad C_{s_2}]\!] \quad [\![\hat{C}_p^1 \quad C_{p_2}^1]\!] \quad \cdots \quad [\![\hat{C}_p^d \quad C_{p_2}^d]\!] \,]$$

Then, a state–space realization for the balanced truncation $\mathcal{G}_{\delta,r}$ of the system $\mathcal{G}_\delta$ is $(A_r, B_r, C_r, D_r; \mathbf{\Delta}_r)$, where

$$\mathbf{\Delta}_r = \{\mathcal{P}\Delta\mathcal{P}^* : \Delta \in \mathbf{\Delta}\}$$

and

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{cccc|c} \hat{A}_{ss} & \hat{A}_{sp}^1 & \cdots & \hat{A}_{sp}^d & \hat{B}_s \\ \hat{A}_{ps}^1 & \hat{A}_{pp}^{11} & \cdots & \hat{A}_{pp}^{1d} & \hat{B}_p^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{A}_{ps}^d & \hat{A}_{pp}^{d1} & \cdots & \hat{A}_{pp}^{dd} & \hat{B}_p^d \\ \hline \hat{C}_s & \hat{C}_p^1 & \cdots & \hat{C}_p^d & D \end{array}\right]$$

with $\mathcal{P}$ being an appropriately defined truncation operator. Namely, we have $\Delta_r = \mathcal{P}\Delta\mathcal{P}^* = \text{diag}(I_{\ell_2}^{r_0}, \hat{\Delta}_1, \ldots, \hat{\Delta}_d)$, where $\hat{\Delta}_i = \text{diag}(\delta_i(0)I_{r_i(0)}, \delta_i(1)I_{r_i(1)}, \ldots)$. Notice that Δ_r is constructed from the same parameters δ_i as those in Δ.

Lemma 10: Suppose $(A, B, C, D; \mathbf{\Delta})$ is a balanced realization of $\mathcal{G}_\delta$. Then the realization of the balanced truncation $\mathcal{G}_{\delta,r}$ is also strongly ℓ_2-stable and balanced.

Proof: To start, there exists a unique permutation Q such that $Q^*\Sigma Q = \text{diag}(\Gamma, \Omega)$ formally; then we have

$$Q^*\tilde{Z}AQ = \begin{bmatrix} \tilde{Z} & \\ & \tilde{Z} \end{bmatrix} \begin{bmatrix} A_r & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \tilde{Z}_2\bar{A}$$
$$Q^*\tilde{Z}B = \begin{bmatrix} \tilde{Z} & \\ & \tilde{Z} \end{bmatrix} \begin{bmatrix} B_r \\ \bar{B}_2 \end{bmatrix} = \tilde{Z}_2\bar{B}$$
$$CQ = [C_r \quad \bar{C}_2] = \bar{C}$$

and $\bar{\Delta} = Q^*\Delta Q = \text{diag}(\Delta_r, \bar{\Delta}_2)$, where $(A_r, B_r, C_r, D_r; \mathbf{\Delta}_r)$ is a realization of the balanced truncation $\mathcal{G}_{\delta,r}$, and the rest of the operators are defined in the obvious way.

As the generalized gramian Σ satisfies both of inequalities (8) and (9), then focusing on (8), and with the aforesaid permutation in mind, the following ensues:

$$\begin{bmatrix} A_r & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \Gamma & \\ & \Omega \end{bmatrix} \begin{bmatrix} A_r^* & \bar{A}_{21}^* \\ \bar{A}_{12}^* & \bar{A}_{22}^* \end{bmatrix} - \begin{bmatrix} \tilde{Z}^*\Gamma\tilde{Z} & \\ & \tilde{Z}^*\Omega\tilde{Z} \end{bmatrix} + \begin{bmatrix} B_r \\ \bar{B}_2 \end{bmatrix} [\, B_r^* \quad \bar{B}_2^* \,] \prec 0.$$

This clearly gives $A_r\Gamma A_r^* - \tilde{Z}^*\Gamma\tilde{Z} + B_rB_r^* \prec 0$. Similarly, starting with (9), we can show that

$$A_r^*\tilde{Z}^*\Gamma\tilde{Z}A_r - \Gamma + C_r^*C_r \prec 0.$$

Thus, directly from the definitions of strong stability and a balanced system, we have the desired conclusion. ■

B. Error Bounds

This subsection gives upper bounds on the error induced in the balanced truncation model reduction process. We start with the following result.

Lemma 11: An NSLPV model $\mathcal{G}_\delta$ is strongly ℓ_2-stable and satisfies the condition $\|G_\delta\| < 1$ for all $\Delta \in \mathbf{\Delta}$ if there exists a positive definite operator V in the commutant of $\mathbf{\Delta}$ such that

$$\begin{bmatrix} -V & 0 & A^* & C^* \\ 0 & -I & B^* & D^* \\ A & B & -\tilde{Z}^*V^{-1}\tilde{Z} & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0. \quad (12)$$

This is a generalization of the sufficiency part of the Kalman–Yakubovich–Popov (KYP) Lemma. Its proof is routine and so we do not include it here. Note that inequality (12) is necessary and sufficient in the purely time-varying case as proved in [15]; however, in our case, it is in general only sufficient.

Theorem 12: Suppose that $(A, B, C, D; \mathbf{\Delta})$ is a balanced realization for the NSLPV system $\mathcal{G}_\delta$, and that the diagonal generalized gramian $\Sigma \in \mathcal{X}$, satisfying both of inequalities (8) and (9), is partitioned as in (10). If $\Omega_i = I_{\ell_2}$ for all $i = 0, 1, \ldots, d$, then the balanced truncation $\mathcal{G}_{\delta,r}$ of $\mathcal{G}_\delta$ satisfies the norm condition

$$\|G_\delta - G_{\delta,r}\| < 2 \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

Proof: As $\mathcal{G}_\delta$ and $\mathcal{G}_{\delta,r}$ are both strongly ℓ_2-stable, then so is $\mathcal{E}_\delta = \{(1/2)(G_\delta - G_{\delta,r}) : \Delta \in \mathbf{\Delta}\}$. One realization of $(1/2)(G_\delta - G_{\delta,r})$ is given in linear fractional form by

$$\frac{1}{2}(G_\delta - G_{\delta,r}) = \begin{bmatrix} \Delta_r & \\ & \bar{\Delta} \end{bmatrix} \star \left[\begin{array}{cc|c} \tilde{Z}A_r & 0 & \frac{1}{\sqrt{2}}\tilde{Z}B_r \\ 0 & \tilde{Z}_2\bar{A} & \frac{1}{\sqrt{2}}\tilde{Z}_2\bar{B} \\ \hline -\frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}\bar{C} & 0 \end{array}\right]$$

where $\bar{A}, \bar{B}, \bar{C}$, and $\bar{\Delta}$ are as defined in the proof of Lemma 10, and $\tilde{Z}_m = \operatorname{diag}(\{J_i\}_{i=1}^m)$ where $J_i = \tilde{Z}$ for all i. In the sequel, we will construct a positive definite operator V that commutes with $\operatorname{diag}(\Delta_r, \bar{\Delta})$ for all $\Delta \in \mathbf{\Delta}$ and satisfies inequality (12) for this $\mathcal{E}_\delta$ realization. Then, invoking Lemma 11 completes the proof.

Given that the diagonal operator $\Sigma \in \mathcal{X}$ satisfies inequalities (8) and (9), then direct applications of the Schur complement formula, along with some permutations, guarantee the validity of the following condition:

$$\begin{bmatrix} -R_1 & K^* \\ K & -\mathcal{Z}_a^*R_2\mathcal{Z}_a \end{bmatrix} \prec 0$$

where $\mathcal{Z}_a = \operatorname{diag}(\tilde{Z}_2, I, \tilde{Z}_2)$,

$$R_i = \begin{bmatrix} \Gamma^{-1} & 0 & 0 & 0 & 0 \\ 0 & \Omega^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_{\ell_2}^{q_i} & 0 & 0 \\ 0 & 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & 0 & \Omega \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & A_r & \bar{A}_{12} \\ 0 & 0 & 0 & \bar{A}_{21} & \bar{A}_{22} \\ 0 & 0 & 0 & C_r & \bar{C}_2 \\ A_r & \bar{A}_{12} & B_r & 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 & 0 & 0 \end{bmatrix}$$

and $q_1 = n_w, q_2 = n_z$. Define invertible operators L and S by

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} -I & 0 & 0 & I & 0 \\ I & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & \sqrt{2}I & 0 & 0 \\ 0 & -I & 0 & 0 & I \end{bmatrix}$$

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & \sqrt{2}I & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix}.$$

Pre- and postmultiplying the previous condition by $\operatorname{diag}(S^*, L)$ and $\operatorname{diag}(S, L^*)$, respectively, give the equivalent inequality

$$\begin{bmatrix} -S^*R_1S & S^*K^*L^* \\ LKS & -\mathcal{Z}_b^*LR_2L^*\mathcal{Z}_b \end{bmatrix} \prec 0 \quad (13)$$

where $\mathcal{Z}_b = \operatorname{diag}(\tilde{Z}_3, I, \tilde{Z})$. Performing the multiplications in this inequality leads to $S^*R_1S =$

$$\begin{bmatrix} \frac{1}{2}(\Gamma^{-1}+\Gamma) & \frac{1}{2}(\Gamma^{-1}-\Gamma) & 0 & 0 & 0 \\ \frac{1}{2}(\Gamma^{-1}-\Gamma) & \frac{1}{2}(\Gamma^{-1}+\Gamma) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) & 0 & \frac{1}{2}(\Omega^{-1}-\Omega) \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}-\Omega) & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) \end{bmatrix}$$

and $LR_2L^* =$

$$\begin{bmatrix} \frac{1}{2}(\Gamma^{-1}+\Gamma) & \frac{1}{2}(\Gamma-\Gamma^{-1}) & 0 & 0 & 0 \\ \frac{1}{2}(\Gamma-\Gamma^{-1}) & \frac{1}{2}(\Gamma^{-1}+\Gamma) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) & 0 & \frac{1}{2}(\Omega-\Omega^{-1}) \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & \frac{1}{2}(\Omega-\Omega^{-1}) & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) \end{bmatrix}$$

$$LKS = \left[\begin{array}{cccc|c} A_r & 0 & 0 & \frac{1}{\sqrt{2}}B_r & \bar{A}_{12} \\ 0 & A_r & \bar{A}_{12} & \frac{1}{\sqrt{2}}B_r & 0 \\ 0 & \bar{A}_{21} & \bar{A}_{22} & \frac{1}{\sqrt{2}}\bar{B}_2 & 0 \\ -\frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}\bar{C}_2 & 0 & -\frac{1}{\sqrt{2}}\bar{C}_2 \\ \hline \bar{A}_{21} & 0 & 0 & \frac{1}{\sqrt{2}}\bar{B}_2 & \bar{A}_{22} \end{array}\right] = \left[\begin{array}{c|c} M & N_{12} \\ \hline N_{21} & \bar{A}_{22} \end{array}\right]$$

Note that, in the preceding expressions, some of the operators might contain at certain time-instants matrices of zero dimensions. In such scenarios, the rows and columns of which the said matrices are elements would not be present, and the corresponding operator inequalities remain valid.

Define the operator V as

$$V = \begin{bmatrix} \frac{1}{2}(\Gamma^{-1}+\Gamma) & \frac{1}{2}(\Gamma^{-1}-\Gamma) & 0 \\ \frac{1}{2}(\Gamma^{-1}-\Gamma) & \frac{1}{2}(\Gamma^{-1}+\Gamma) & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) \end{bmatrix}.$$

Observe that, since $S^* R_1 S \succ 0$, then $V \succ 0$. Also, V clearly commutes with the operator $\mathrm{diag}(\Delta_r, \bar{\Delta})$. Recall that, by assumption, $\Omega = I$; hence, $(1/2)(\Omega^{-1}+\Omega) = I$ and $(1/2)(\Omega^{-1}-\Omega) = 0$. With this in mind, it is not difficult to see that inequality (13) implies that

$$\begin{bmatrix} -\begin{bmatrix} V & \\ & I_{\ell_2}^{n_w} \end{bmatrix} & M^* \\ M & -\begin{bmatrix} \tilde{Z}_3^* V^{-1} \tilde{Z}_3 & \\ & I_{\ell_2}^{n_z} \end{bmatrix} \end{bmatrix} \prec 0.$$

Then, invoking Lemma 11, we get $\|(1/2)(G_\delta - G_{\delta,r})\| < 1$ for all $\Delta \in \mathbf{\Delta}$. ■

Theorem 13: Given a strongly ℓ_2-stable NSLPV model $\mathcal{G}_\delta$, then its balanced truncation $\mathcal{G}_{\delta,r}$ satisfies

$$\|G_\delta - G_{\delta,r}\| < 2\sum_{i=0}^{d}\sum_{j}\omega_{i,j}, \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

where $\omega_{i,j}$ are the distinct diagonal entries of the block-diagonal operator Ω_i.

The proof follows from scaling, Lemma 10, and repeated application of the previous theorem. Note that this error bound might involve an infinite summation which in general may not converge to a finite number. In the following, we improve on this result and derive tighter bounds. We will first consider balanced systems where the singular values corresponding to the states and parameters to be truncated are monotonic in time. Before doing this, it will be convenient to establish the following terminology.

Definition 14: Given a scalar sequence α_k defined on a subset $\mathcal{W}$ of the nonnegative integers, we define the following hold rule which extends the domain of α_k to all $k \geq 0$:

Let $k_{\min} = \min\{k \geq 0 : k \in \mathcal{W}\}$ and then set

$$\alpha_k = \begin{cases} \alpha_{k_{\min}}, & \text{if } 0 \leq k \leq k_{\min} \\ \alpha_q, \text{ where } q := \max\{q \leq k : q \in \mathcal{W}\}, & \text{if } k_{\min} < k. \end{cases}$$

We now have the following result.

Theorem 15 (Monotonic Case): Suppose that $(A, B, C, D; \mathbf{\Delta})$ is a balanced realization for the NSLPV system $\mathcal{G}_\delta$, and that the diagonal generalized gramian $\Sigma \in \mathcal{X}$, satisfying both of inequalities (8) and (9), is partitioned as in (10). Also, suppose that, for all $i = 0, 1, \ldots, d$ and $k \in \mathcal{F}_i = \{k \geq 0 : n_i(k) \neq r_i(k)\}$, we have $\Omega_i(k) = \omega_{i,k} I_{s_i(k)}$, where $s_i(k) = n_i(k) - r_i(k)$ and the sequence of scalars $\omega_{i,k}$ for all $k \in \mathcal{F}_i$ is monotonic. Then, the balanced truncation $\mathcal{G}_{\delta,r}$ of $\mathcal{G}_\delta$ satisfies the norm condition

$$\|G_\delta - G_{\delta,r}\| < 2\sum_{i=0}^{d}\sup_{k \in \mathcal{F}_i}\omega_{i,k} \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

Proof: It is sufficient to prove the theorem for the case where only one parameter or state block is being truncated (i.e., $s_i = 0$ for all i except for one, say $j \in \{0, 1, 2, \ldots, d\}$), since the general case then follows simply by the standard use of the telescoping series and triangle inequality. Also, we assume without loss of generality that $\omega_{j,k} \leq 1$ for all $k \in \mathcal{F}_j$; this can always be achieved by scaling inequalities (8) and (9).

To begin, we extend the domain of definition of $\omega_{j,k}$ to all $k \geq 0$ using the hold rule defined in Definition 14; note that the extended sequence is still monotonic. We now split the remainder of our proof into two separate cases, one where this sequence is nondecreasing and the other where it is nonincreasing.

Case $\omega_{j,k}$ nondecreasing:

In this case, we have $\omega_{j,k} \leq \omega_{j,k+1}$ for all $k \geq 0$. We define the state space transformation $T \in \mathcal{T}$ as

$$[\![T]\!]_k = (\omega_{j,k})^{-\frac{1}{2}} I. \tag{14}$$

Note that, since $\Sigma \succ 0$, then T is indeed bounded. This gives the following balanced realization for $\mathcal{G}_\delta$: $(\bar{A}, \bar{B}, \bar{C}, D; \mathbf{\Delta}) :=$

$$((\tilde{Z}^* T \tilde{Z}) A T^{-1}, (\tilde{Z}^* T \tilde{Z}) B, C T^{-1}, D; \mathbf{\Delta}). \tag{15}$$

For convenient reference, we will use $\bar{\mathcal{G}}_\delta$ to refer to the system $\mathcal{G}_\delta$ when the realization in use is (15).

Our goal now is to show that this new realization is balanced. To this end, given the state transformation T, we use (8) and (9) to get

$$\begin{aligned} &\bar{A}\bar{\Sigma}\bar{A}^* - \tilde{Z}^*\bar{\Sigma}\tilde{Z} + \bar{B}\bar{B}^* \prec 0 \\ &\bar{A}^*\tilde{Z}^*((T^*)^{-1}\Sigma T^{-1})\tilde{Z}\bar{A} - (T^*)^{-1}\Sigma T^{-1} + \bar{C}^*\bar{C} \prec 0 \end{aligned} \tag{16}$$

where $\bar{\Sigma} = T\Sigma T^*$. Because of the special structure of T and the fact that $\omega_{j,k} \leq \omega_{j,k+1} \leq 1$, it is not difficult to see that

$$\begin{aligned} \bar{C}^*\bar{C} &= (T^*)^{-1}C^*CT^{-1} \preceq T^*C^*CT \\ \bar{A}^*\tilde{Z}^*\bar{\Sigma}\tilde{Z}\bar{A} &= \bar{A}^*\tilde{Z}^*(T\Sigma T^*)\tilde{Z}\bar{A} \\ &\preceq (T^*)^2\bar{A}^*\tilde{Z}^*((T^*)^{-1}\Sigma T^{-1})\tilde{Z}\bar{A}T^2. \end{aligned}$$

Then, pre- and post-multiplying inequality (16) by $(T^*)^2$ and T^2 respectively and then using the above inequalities give

$$\bar{A}^*\tilde{Z}^*\bar{\Sigma}\tilde{Z}\bar{A} - \bar{\Sigma} + \bar{C}^*\bar{C} \prec 0.$$

Hence, $\bar{\Sigma}$ is a diagonal gramian satisfying the generalized Lyapunov inequalities for the system realization $\bar{\mathcal{G}}_\delta$. Notice that, by the definition of T, we have

$$\begin{aligned} \bar{\Sigma}_j(k) &= T_j(k)\Sigma_j(k)T_j^*(k) = (\omega_{j,k})^{-1}\begin{bmatrix} \Gamma_j(k) & \\ & \Omega_j(k) \end{bmatrix} \\ &= \begin{bmatrix} (\omega_{j,k})^{-1}\Gamma_j(k) & \\ & I \end{bmatrix} = \begin{bmatrix} \bar{\Gamma}_j(k) & \\ & \bar{\Omega}_j(k) \end{bmatrix}. \end{aligned}$$

Thus, $\bar{\Omega}_j = I_{\ell_2}$, and so, by invoking Theorem 12, we deduce that the balanced truncation $\bar{\mathcal{G}}_{\delta,r}$ of the system $\bar{\mathcal{G}}_\delta$ satisfies the norm condition

$$\|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2 \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

Now, because of the special structure of T, it is not difficult to see that, for each $\Delta \in \mathbf{\Delta}$, the error system realizations $G_\delta - G_{\delta,r}$ and $\bar{G}_\delta - \bar{G}_{\delta,r}$ are in fact equivalent, and as a result, we have

$$\|G_\delta - G_{\delta,r}\| = \|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2.$$

Case $\omega_{j,k}$ nonincreasing:

A similar argument applies where here the state transformation $T \in \mathcal{T}$ is defined as $[\![T]\!]_k = (\omega_{j,k})^{\frac{1}{2}} I$. ■

We now consider the more general case, where singular values need not be monotonic in time. But first, we require the following definition from [5].

Definition 16: Given a vector $v = (v_1, v_2, \ldots, v_s)$ for some integer $s \geq 1$, suppose that v_1 cannot be considered as a local maximum and v_s cannot be considered as a local minimum. Then vector v has m local maxima $v_{\max,i}$ and m local minima $v_{\min,i}$ for some integer $m \geq 0$, and the max-min ratio of v, denoted $\mathcal{S}_v$, is defined as

$$\mathcal{S}_v = v_1 \prod_{i=1}^{m} \frac{v_{\max,i}}{v_{\min,i}}, \qquad m > 0$$
$$\mathcal{S}_v = v_1, \qquad m = 0.$$

Theorem 17 (Nonmonotonic Case): Given a balanced realization $(A, B, C, D; \mathbf{\Delta})$ for the NSLPV system $\mathcal{G}_\delta$, suppose that a diagonal operator $\Sigma \in \mathcal{X}$ satisfies both of inequalities (8) and (9) and is partitioned as in (10), where, for all $i = 0, 1, \ldots, d, \Omega_i(k) = \omega_{i,k} I_{s_i(k)}$, with $s_i(k) = n_i(k) - r_i(k)$. Define the vector $\hat{\omega}_i$ to consist of the elements $\omega_{i,k}$ for all time k such that $s_i(k)$ is nonzero. If for each $i = 0, 1, \ldots, d$ we have $\dim(\hat{\omega}_i) < \infty$, then the balanced truncation $\mathcal{G}_{\delta,r}$ of $\mathcal{G}_\delta$ satisfies the norm condition

$$\|G_\delta - G_{\delta,r}\| < 2 \sum_{i=0}^{d} \mathcal{S}_{\hat{\omega}_i} \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

The proof that follows uses the same idea as that of Theorem 15. Basically, we need to define some state space transformation that results in a balanced realization for the system $\mathcal{G}_\delta$ where the diagonal gramian $\bar{\Sigma}$ solving the Lyapunov inequalities for this realization is such that $\bar{\Omega}_j = I_{\ell_2}$. Then, invoking Theorem 12 completes the proof. The choice of the state space transformation used is inspired by that of the monotonic case.

Proof: As with the proof of Theorem 15 it is sufficient to prove the result for the case where the only Ω_i that has non-zero dimension is Ω_j for some fixed j in $\{0, 1, \ldots, d\}$; without loss of generality we assume that $\omega_{j,k} \leq 1$ for all k.

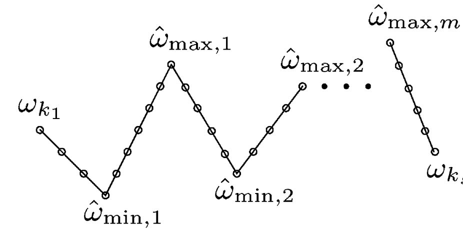
To keep the notation simple, we suppress the subscript j in $\omega_{j,k}$ and $\hat{\omega}_j$. The vector $\hat{\omega}$ is of the form

$$\hat{\omega} = (\omega_{k_1}, \omega_{k_2}, \ldots, \hat{\omega}_{\min,1}, \ldots, \hat{\omega}_{\max,1}, \ldots, \hat{\omega}_{\max,m}, \ldots, \omega_{k_s})$$

corresponding to values of the sequence ω_k evaluated at the ordered time points

$$(k_1, k_2, \ldots, k_{\min,1}, \ldots, k_{\max,1}, \ldots, k_{\max,m}, \ldots, k_s).$$

The denoted local minima and maxima of the vector $\hat{\omega}$ are as defined in Definition 16. We now use the hold rule of Definition 14 to extend the sequence ω_k to all $k \geq 0$; the maxima and minima of ω_k are illustrated as follows:



We define the state–space transformation $T \in \mathcal{T}$ as

$$[\![T]\!]_k = \begin{cases} \omega_{k_1}^{\frac{1}{2}} I, & \text{for } k = 0, 1, \ldots, k_1 - 1 \\ \omega_k^{\frac{1}{2}} I, & \text{for } k = k_1, k_1 + 1, \ldots, k_{\min,1} \\ \hat{\omega}_{\min,1} \omega_k^{-\frac{1}{2}} I, & \text{for } k = k_{\min,1} + 1, \ldots, k_{\max,1} \\ \hat{\omega}_{\min,1} \hat{\omega}_{\max,1}^{-1} \omega_k^{\frac{1}{2}} I, & \text{for } k = k_{\max,1} + 1, \ldots, k_{\min,2} \\ \vdots & \vdots \\ \rho \omega_k^{\frac{1}{2}} I, & \text{for } k = k_{\max,m} + 1, \ldots, k_s \\ \rho \omega_{k_s}^{\frac{1}{2}} I, & \text{for } k = k_s + 1, k_s + 2, \ldots \end{cases}$$

where $\rho = \prod_{i=1}^{m} \hat{\omega}_{\min,i} \hat{\omega}_{\max,i}^{-1}$. Also, define operators $P, Q \in \mathcal{T}$ such that $[\![P]\!]_k = \omega_k^{1/2} I$ and $Q = TP^{-1}$. It is not difficult to see that the constituent scalars of operator T define a nonincreasing sequence, and so do those of operator Q and those of operator QP^2. Then, given the equivalent realization

$$(\bar{A}, \bar{B}, \bar{C}, D; \mathbf{\Delta}) = ((\tilde{Z}^* T \tilde{Z}) A T^{-1}, (\tilde{Z}^* T \tilde{Z}) B, C T^{-1}, D; \mathbf{\Delta})$$

of the system $\mathcal{G}_\delta$, which we denote for ease of reference by $\bar{\mathcal{G}}_\delta$, and because of the special structure of T and the assumption that $\omega_{j,k} \leq 1$, the following ensue:

$$\bar{A}\bar{\Sigma}\bar{A}^* - \tilde{Z}^* \bar{\Sigma} \tilde{Z} + P^{-2} Q^{-1} \bar{B} \bar{B}^* (Q^*)^{-1} (P^*)^{-2} \prec 0$$
$$\bar{A}^* \tilde{Z}^* \bar{\Sigma} \tilde{Z} \bar{A} - \bar{\Sigma} + Q^* \bar{C}^* \bar{C} Q \prec 0$$

where $\bar{\Sigma} = (P^*)^{-1} \Sigma P^{-1}$. Notice that $P^{-2} Q^{-1} \succeq \omega_{k_1}^{-1} I$ and $Q \succeq \rho I$. Thus, the diagonal operator $\bar{\Sigma}$ satisfies the generalized Lyapunov inequalities (8) and (9) for the realization $(\bar{A}, \omega_{k_1}^{-1} \bar{B}, \rho \bar{C}, D; \mathbf{\Delta})$. As $\bar{\Omega}_j = I_{\ell_2}$, then, invoking Theorem 12, we get

$$\omega_{k_1}^{-1} \rho \|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2, \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

Finally, the special structure of operator T and the fact that $\mathcal{S}_{\hat{\omega}} = (\omega_{k_1}^{-1} \rho)^{-1}$ lead to

$$\|G_\delta - G_{\delta,r}\| = \|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2\mathcal{S}_{\hat{\omega}}, \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

■

We remark that Theorem 13 generalizes the LTV result in [4] to the NSLPV framework. Also, Theorems 15 and 17 are mainly generalizations of their LTV counterparts in [5], with the important exception that the truncations in the theorems need not be restricted to connected intervals. To illustrate how to apply these results, we consider the following hypothetical example. Suppose we are to truncate the states corresponding to the sequence $\Omega_0(k) = \omega_{0,k} I_{s_0(k)}$ for $k \in [1, 9]$, where

$$\{\omega_{0,k}\}_{k=1}^{9} = \{1, 0.75, 2, 1.25, 3, 1.75, 4, 2.25, 5\}.$$

Then the corresponding error bound obtained from Theorem 13 is

$$2 \times (1 + 0.75 + 2 + 1.25 + 3 + 1.75 + 4 + 2.25 + 5) = 42.$$

This is exactly the same bound that the main result of [4] would give assuming a standard LTV system. If we are to apply Theorem 17 to truncate the states in one step, then we obtain the error bound

$$2 \times 1 \times \frac{2}{0.75} \times \frac{3}{1.25} \times \frac{4}{1.75} \times \frac{5}{2.25} \approx 65.$$

This bound is quite conservative and can be significantly improved if we truncate the states in three steps and accordingly divide the sequence $\omega_{0,k}$ into the following: $\{1, 0.75, 2, 1.25\}$, $\{3, 1.75, 4, 2.25\}$, and $\{5\}$. Then, applying Theorem 17 recursively, we obtain the improved error bound

$$2 \times \left(1 \times \frac{2}{0.75} + 3 \times \frac{4}{1.75} + 5\right) \approx 29.$$

This can also be obtained from the results of [5] if the system in question is a standard LTV system. But, in our case, we can actually further improve on the last bound by dividing the sequence $\omega_{0,k}$ into the two monotonic sequences $\{1, 2, 3, 4, 5\}$ and $\{0.75, 1.25, 1.75, 2.25\}$ and then applying Theorem 15 twice to get the error bound $2 \times (5 + 2.25) = 14.5$.

Finally, it worth noting that we can generalize the LTV error bound result of [16] to the NSLPV framework, but we will refrain from doing so here due to space considerations. This result is very useful when truncating lots of states over small time intervals.

C. Eventually Periodic LPV Systems

This subsection focuses on the balanced truncation of *eventually periodic* LPV systems. These systems are aperiodic for an initial amount of time, and then become periodic afterwards. One scenario in which they originate is when parameterizing nonlinear systems about eventually periodic trajectories. Such trajectories can be arbitrary for a finite amount of time, but then settle down into a periodic orbit; a special case of this occurs when a system transitions between two operating points. In addition to that, eventually periodic systems naturally arise when considering problems involving plants with uncertain initial states. Note that both finite horizon and periodic systems are subclasses of eventually periodic systems. We refer the reader to [17]–[19] for some useful results on eventually periodic models. We now give a precise definition of an eventually periodic operator.

Definition 18: A block-diagonal mapping P on ℓ_2 is (h, q)-eventually periodic if, for some integers $h \geq 0, q \geq 1$, we have

$$Z^q((Z^*)^h P Z^h) = ((Z^*)^h P Z^h) Z^q$$

that is P is q-periodic after an initial transient behavior up to time h. Moreover, a partitioned operator, whose elements are block-diagonal, is (h, q)-eventually periodic if each of its block-diagonal elements is (h, q)-eventually periodic.

Theorem 19: Suppose that state space operators A, B, and C are (h, q)-eventually periodic. Then solutions $X, Y \in \mathcal{X}$ satisfying Lyapunov inequalities (8) and (9) exist if and only if (h, q)-eventually periodic solutions $X_{\text{eper}}, Y_{\text{eper}} \in \mathcal{X}$ exist.

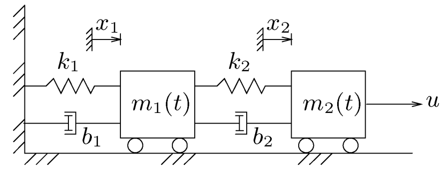


Fig. 2. Translational system.

The outline of the proof is as follows: first, employ a similar averaging technique to that used in [14] to show that the periodic part of any of the generalized Lyapunov inequalities admits a q-periodic solution if feasible, then, having established that, the above result follows from scaling.

Thus, if the system is strongly ℓ_2-stable and (h, q)-eventually periodic, then we can construct an (h, q)-eventually periodic balanced realization with an (h, q)-eventually periodic diagonal gramian $\Sigma \in \mathcal{X}$ satisfying Lyapunov inequalities (8) and (9).

Theorem 20: Suppose that system $\mathcal{G}_\delta$ is an (h, q)-eventually periodic system with a balanced realization $(A, B, C, D; \mathbf{\Delta})$. Then the following hold.

i) There exists an (h, q)-eventually periodic diagonal operator $\Sigma \in \mathcal{X}$, partitioned as in (10), satisfying both of the generalized Lyapunov inequalities (8) and (9);

ii) The balanced truncation $\mathcal{G}_{\delta,r}$ of $\mathcal{G}_\delta$ is balanced and satisfies the finite error bound

$$\|G_\delta - G_{\delta,r}\| < E_{fh} + 2\sum_{i=0}^{d}\sum_{j}\omega_{i,j} < \infty$$

for all $\Delta \in \mathbf{\Delta}$,where $\omega_{i,j}$ are the *distinct* diagonal entries of the matrix $\text{diag}(\Omega_i(h), \ldots, \Omega_i(h+q-1))$, and E_{fh} is the finite upper bound on the error induced in the balanced truncation of the finite horizon part of $\mathcal{G}_\delta$ and is derived by applying Theorem 17.

Remark 21: Given an eventually periodic system, suppose that we are to truncate states or parameters over the whole period and that the relevant singular values are mostly distinct. Then clearly, the larger the period length is, the less useful the error bounds become since computing these bounds involves summations over the period. In this case, it might be more plausible to solve for controllability and observability gramians which are *constant* over the period; this would ensure that the singular values are equal over the period and hence eliminate the need for large summations when computing the associated error bounds.

V. Example

Consider the two-mass translational system in Fig. 2. The state variables x_1 and x_2 denote the positions of masses m_1 and m_2, respectively. The masses are *time-varying* and are assumed to move horizontally on frictionless bearings. The springs are undeflected when $x_1 = x_2 = 0$. The control input is the horizontal force u applied to mass m_2. Suppose that the spring constants $k_1 = 10^4(1 + x_1^4), k_2 = 100$, and the friction coefficients $b_1 = 100(1 + \dot{x}_1^4), b_2 = 10$. We assume that the masses exhibit eventually time-invariant variation, namely each starting at an initial value of 50 and then decreasing linearly with time to a final permanent value of 25 after $t = 10$. Suppose that the measurable output is only position x_2, then the continuous-time state–space realization of this system is given by

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}(x_1, \dot{x}_1, t)x(t) + \mathcal{B}(t)u(t) \\ y(t) &= \mathcal{C}x(t) \end{aligned} \tag{17}$$

where the state $x = (x_1, x_2, \dot{x}_1, \dot{x}_2), \mathcal{C} = [0\ 1\ 0\ 0]$, $\mathcal{B}(t) = [0\ 0\ 0\ (1/m_2)]^*$, and

$$\mathcal{A}(x_1, \dot{x}_1, t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{b_1+b_2+\dot{m}_1}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2+\dot{m}_2}{m_2} \end{bmatrix}.$$

We now derive an NSLPV model $\mathcal{G}_\delta$ that captures the nonlinear dynamics of this translational system. To begin, one obvious choice for parametrization of the above system equations is $\delta_1 = x_1, \delta_2 = \dot{x}_1$. The next step is to formulate the state–space equation (17) in an LFT framework, namely

$$\begin{aligned} \dot{x}(t) &= A^c_{ss}(t)x(t) + A^c_{sp}(t)\beta_c(t) + B^c_s(t)u(t) \\ \alpha_c(t) &= A^c_{ps}(t)x(t) + A^c_{pp}(t)\beta_c(t) + B^c_p u(t) \\ \beta_c(t) &= \mathrm{diag}\,(\delta_1(t)I, \delta_2(t)I)\,\alpha_c(t). \end{aligned}$$

In other words, we need to write the matrix-valued functions $\mathcal{A}(\delta_1, \delta_2, t)$ and $\mathcal{B}(t)$ as

$$\begin{aligned} \mathcal{A}(\delta_1, \delta_2, t) &= \underline{\Delta}(t) \star \left[\begin{array}{c|c} A^c_{pp}(t) & A^c_{ps}(t) \\ \hline A^c_{sp}(t) & A^c_{ss}(t) \end{array}\right] \\ \mathcal{B}(t) &= \underline{\Delta}(t) \star \left[\begin{array}{c|c} A^c_{pp}(t) & B^c_p(t) \\ \hline A^c_{sp}(t) & B^c_s(t) \end{array}\right]. \end{aligned} \tag{18}$$

Since $\mathcal{B}(t)$ only has explicit dependence on time t, then clearly B^c_p has to be zero and $B^c_s(t) = \mathcal{B}(t)$. Also, as $\mathcal{A}(\delta_1, \delta_2, t)$ has polynomial dependence on the parameters, the corresponding LFT formulation is straightforward, and uses mainly the following:

$$a_0 + a_1\delta + \cdots + a_n\delta^n = \delta I_n \star \left[\begin{array}{cccc|c} 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ \hline a_1 & \cdots & a_{n-1} & a_n & a_0 \end{array}\right].$$

Then, setting

$$\begin{aligned} \mathcal{P}_1 &= [0\ 0\ 0\ -(10^4/m_1)] \\ \mathcal{P}_2 &= [0\ 0\ 0\ -(100/m_1)] \\ \mathcal{P}_3 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathcal{P}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

it is not difficult to see that

$$A^c_{ss} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{10^4+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{100+b_2+\dot{m}_1}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2+\dot{m}_2}{m_2} \end{bmatrix}$$

$$A^c_{sp} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathcal{P}_1 & \mathcal{P}_2 \\ 0 & 0 \end{bmatrix} \quad A^c_{ps} = \begin{bmatrix} \mathcal{P}_3 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{P}_3 & 0 \end{bmatrix}, \quad A^c_{pp} = \begin{bmatrix} \mathcal{P}_4 & 0 \\ 0 & \mathcal{P}_4 \end{bmatrix}.$$

We now discretize the continuous-time LFT model. In order to simplify this discretization, we choose a sufficiently small sampling period τ, namely $\tau = 0.1$, so that it is reasonable to assume that, for all discrete time-instants $k \geq 0$, the scheduled parameters (δ_1, δ_2) vary very slowly in the time interval $[k\tau, (k+1)\tau)$ that their values on this interval can be approximated by $\delta_i(k) = \delta_i(k\tau)$ for $i = 1, 2$. Then, we can use zero-order hold sampling to obtain the following discrete-time state–space equation:

$$x(k+1) = A_{ss}(k)x(k) + A_{sp}(k)\beta(k) + B_s(k)u(k)$$

where we have $x(k) = x(k\tau), \beta(k) = \beta_c(k\tau), A_{ss}(k) = \Phi_{ss}((k+1)\tau, k\tau)$, $\Phi_{ss}(\cdot,\cdot)$ being the state transition matrix associated with the A-matrix $A^c_{ss}(t)$, and

$$\begin{aligned} A_{sp}(k) &= \int_{k\tau}^{(k+1)\tau} \Phi_{ss}((k+1)\tau, s)A^c_{sp}(s)\,ds \\ B_s(k) &= \int_{k\tau}^{(k+1)\tau} \Phi_{ss}((k+1)\tau, s)B^c_s(s)\,ds. \end{aligned}$$

Alternatively, as proposed in [20], we can use a bilinear transformation to obtain a discrete-time trapezoidal approximation of the continuous-time LFT plant.

At this point, we assume that the parameters δ_1, δ_2 are such that $|\delta_1|, |\delta_1| \leq (1/5)$. Then, this bound is absorbed into the plant so that the new scaled parameters $\bar{\delta}_i$ satisfy $|\bar{\delta}_i| \leq 1$, where $\delta_i = (1/5)\bar{\delta}_i$ for $i = 1, 2$. Also, due to this scaling, we get $A_{ps} = (1/5)A^c_{ps}, A_{pp} = (1/5)A^c_{pp}, B_p = (1/5)B^c_p$. As a result, we obtain the following discrete-time (100, 1)-eventually periodic LPV model:

$$\begin{bmatrix} x(k+1) \\ \alpha(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A_{ss}(k) & A_{sp}(k) & B_s(k) \\ A_{ps} & A_{pp} & B_p \\ C_s & C_p & D \end{bmatrix} \begin{bmatrix} x(k) \\ \beta(k) \\ u(k) \end{bmatrix}$$
$$\beta(k) = \mathrm{diag}(\bar{\delta}_1(k)I_4, \bar{\delta}_2(k)I_4)\alpha(k) \quad |\bar{\delta}_i(k)| \leq 1 \tag{19}$$

where $u \in \ell_2, x(0) = 0, C_s = \mathcal{C}$, and C_p, D are zero matrices.

This system is strongly ℓ_2-stable since the corresponding Lyapunov strict inequalities admit positive definite solutions. Since the error bound is clearly dependent on these solutions, we need to make sure that such solutions satisfy some criterion that yields reasonable error bounds, for example, selecting the solution with the minimum trace. Hence, we solve the following semidefinite programming optimization problem:

$$\begin{aligned}
&\text{minimize:} \sum_{k=0}^{k=100}\sum_{i=0}^{i=2}(\text{trace } X_i(k) + \text{trace } Y_i(k))\\
&\text{subject to:}\\
&[\![A]\!]_k \text{ diag } (X_0(k), X_1(k), X_2(k))[\![A]\!]_k^*\\
&\quad - \text{diag } (X_0(k+1), X_1(k), X_2(k)) + [\![B]\!]_k[\![B]\!]_k^* \prec 0\\
&[\![A]\!]_k^* \text{ diag } (Y_0(k+1), Y_1(k), Y_2(k))[\![A]\!]_k\\
&\quad - \text{diag } (Y_0(k), Y_1(k), Y_2(k)) + [\![C]\!]_k^*[\![C]\!]_k \prec 0\\
&X_i(k), Y_i(k) \succ 0 \quad \text{for all} \quad i = 0, 1, 2 \text{ and } k = 0, 1, \dots, 100\\
&\text{with } X_0(101) = X_0(100), \quad Y_0(101) = Y_0(100).
\end{aligned}$$

Let $X_i(k) = P_i^*(k)P_i(k)$ and $Y_i(k) = Q_i^*(k)Q_i(k)$ be the Cholesky factorizations of the generalized gramians for all $i = 0, 1, 2$ and $k = 0, 1, \dots, 100$. Then, performing a singular value decomposition on $Q_i(k)P_i^*(k)$, namely $Q_i(k)P_i^*(k) = U_i(k)\Sigma_i(k)V_i^*(k)$, we compute the balancing state transformation matrix $T_i(k)$ and its inverse as

$$T_i(k) = \Sigma_i^{-\frac{1}{2}}(k)U_i^*(k)Q_i(k)$$
$$T_i^{-1}(k) = P_i^*(k)V_i(k)\Sigma_i^{-\frac{1}{2}}(k)$$

and consequently obtain the balanced realization $(\bar{A}, \bar{B}, \bar{C})$, where

$$\begin{aligned}
[\![\bar{A}]\!]_k &= \text{diag}(T_0(k+1), T_1(k), T_2(k))[\![A]\!]_k\\
&\quad \times \text{diag}\left(T_0^{-1}(k), T_1^{-1}(k), T_2^{-1}(k)\right)\\
[\![\bar{B}]\!]_k &= \text{diag}(T_0(k+1), T_1(k), T_2(k))[\![B]\!]_k\\
\text{and } [\![\bar{C}]\!]_k &= [\![C]\!]_k \text{ diag}\left(T_0^{-1}(k), T_1^{-1}(k), T_2^{-1}(k)\right)
\end{aligned}$$

for $k = 0, 1, \dots, 100$, with $T_0(101) = T_0(100)$.

All computations are carried out in Matlab 7.0. We use SeDuMi [21], along with the interface [22], to solve the various semidefinite programming optimization problems in this example. As for tuning the solver parameters, we set the precision level equal to 10^{-12}, and, since SeDuMi regards all LMI constraints as nonstrict inequalities, we add, when necessary, sufficiently small constant positive definite terms to the LMIs to ensure strict inequalities and, more importantly, positive definite gramians.

The singular values comprising the generalized diagonal gramian Σ are plotted in Fig. 3. Notice that, at each time instant k, the singular values corresponding to the four copies of parameter δ_1 cannot be distinguished on the plot, and are in fact numerically almost identical. We will exploit this situation and set the four singular values at each point in time equal to their maximum, which is the one corresponding to the first copy of δ_1. The same can be said about the singular values associated with the parameter δ_2. Now, it is not difficult to see from the Lyapunov inequalities that the norm of the A-operator of a balanced NSLPV realization is always less than one; specifically, in this example, we have $\|\bar{A}\| < 1$. Then, to ensure the strict feasibility of the Lyapunov inequalities, we make use of the fact that $\bar{A}^*\bar{A} \prec I$ and add to the modified gramian, which we denote by $\tilde{\Sigma}$, a sufficiently small positive definite term ϵI so

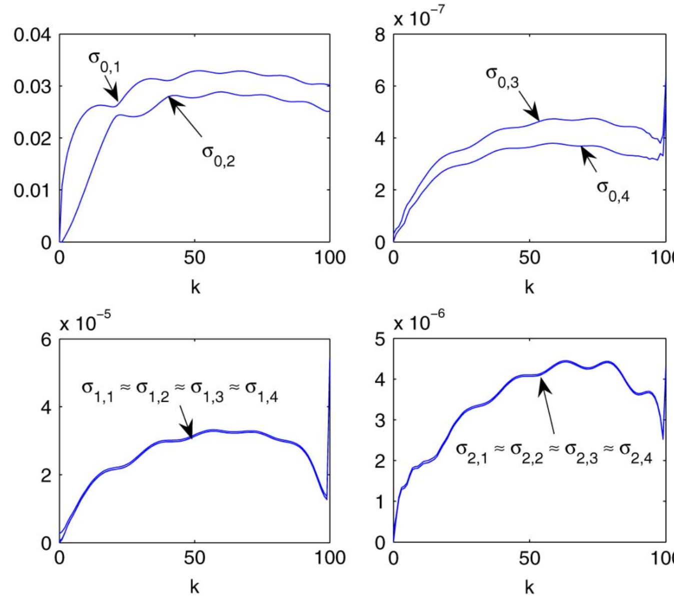


Fig. 3. $\Sigma_i(k) = \text{diag}(\sigma_{i,1}(k), \sigma_{i,2}(k), \sigma_{i,3}(k), \sigma_{i,4}(k))$ for $i = 0, 1, 2$.

that the positive definite diagonal operator $\tilde{\Sigma} + \epsilon I$ satisfies the strict Lyapunov inequalities. The value for ϵ that we use here is 1.5×10^{-7}, and the operator $\tilde{\Sigma} + \epsilon I$ is the gramian used when computing the error bounds. As a result, if we are to truncate the four copies of parameter δ_1 or δ_2, then we only need to account for the singular values corresponding to the first copy when calculating the error bound.

Given the setup of this problem, it is clear that we can truncate both parameters as well as two of the states of the balanced system over all time instants without inducing any significant error. The question is whether this can be reflected in the corresponding error bound given by the results of Section IV-B. It is obvious from the discussion at the end of Section IV-B that the aforesaid results generally give different bounds depending on how the theorems are applied. This is not an issue in the case of systems with small finite horizons since, in such cases, simple algorithms can be written to get the tightest bounds possible. However, calculating useful error bounds can become quite challenging when dealing with large finite horizons. This evokes a very interesting research problem which is worth looking into in future work, namely developing a fast computational algorithm that effectively applies the results of Section IV-B to calculate useful bounds. For this example, we will simply consider each parameter and each state to be truncated one at a time, then implement the truncation over the finite horizon in one step, and apply Theorem 17 to compute the error bound. As for the truncation over the time-invariant part, the corresponding error bound is given by the now standard "twice the sum of the tail" formula. Note that, since we assume zero initial conditions, we can truncate the states and parameters at $k = 0$ without inducing any error and, hence, we need not account for the singular values corresponding to time $k = 0$ when computing the error bound.

With this said, truncating the four copies of parameter δ_2, the four copies of parameter δ_1, and the fourth and third states of the balanced system over all time instants results in the error bounds 1.8551×10^{-5}, 1.7569×10^{-4}, and 5.3885×10^{-6}, respectively.

Thus, the overall bound on the error induced in such a truncation is

$$E = 1.8551 \times 10^{-5} + 1.7569 \times 10^{-4} + 5.3885 \times 10^{-6} = 1.9963 \times 10^{-4}.$$

Note that the gramians and hence the error bounds depend on the solver used. For instance, if we are to use the solver mincx, given in the Robust Control Toolbox, with default settings, then the corresponding error bound would be 47.412×10^{-4}, roughly 24 times the bound given by SeDuMi. In addition to this disadvantage, it is worth noting that, even if we prevent switching to QR factorization, mincx remains very slow compared to SeDuMi especially when it comes to large finite horizons.

At this point, we would like to see if E actually constitutes a good upper bound on the worst case error for such a reduction. For that matter, we first need to compute the worst case error, which is defined as $\sup_\Delta \|G_\delta - G_{\delta,r}\|$. It is not necessary to determine the exact value of this supremum; rather, it is sufficient to find good lower and upper bounds on this value, from which we can deduce an estimate. To begin, an upper bound on this supremum can be provided by an algorithm based on the KYP lemma, namely $\sup_\Delta \|G_\delta - G_{\delta,r}\| < \sqrt{\gamma_{\text{opt}}}$, where γ_{opt} is the solution of the following convex optimization problem:

minimize: γ

subject to:

$$\left[\!\left[\begin{matrix} A_e & B_e \\ C_e & 0 \end{matrix}\right]\!\right]_k^* \left[\!\left[\begin{matrix} \tilde{Z}^* R \tilde{Z} & \\ & I \end{matrix}\right]\!\right]_k \left[\!\left[\begin{matrix} A_e & B_e \\ C_e & 0 \end{matrix}\right]\!\right]_k - \left[\!\left[\begin{matrix} R & \\ & \gamma I \end{matrix}\right]\!\right]_k \prec 0$$

$[\![R]\!]_k = \operatorname{diag}(R_0(k), R_1(k), R_2(k)) \succ 0$

for all $k = 0, 1, \ldots, 100$, with $R_0(101) = R_0(100)$.

Note that $(A_e, B_e, C_e; \Delta_e)$ is a realization for the error system $G_\delta - G_{\delta,r}$, where Δ_e in this case is given by $[\![\Delta_e]\!]_k = \operatorname{diag}(I_6, \bar{\delta}_1 I_4, \bar{\delta}_2 I_4)$, and the (100, 1)-eventually periodic operators A_e, B_e, C_e are obtained from the balanced system operators $\bar{A}, \bar{B}, \bar{C}$ as follows:

$$[\![A_e]\!]_k = \begin{bmatrix} P\bar{A}_{ss}(k)P^* & 0 & 0 \\ 0 & \bar{A}_{ss}(k) & [\![\bar{A}_{sp}]\!]_k \\ 0 & [\![\bar{A}_{ps}]\!]_k & [\![\bar{A}_{pp}]\!]_k \end{bmatrix}$$

$$[\![C_e]\!]_k = [-\bar{C}_s(k)P^* \quad \bar{C}_s(k) \quad [\![C_p]\!]_k]$$

$$[\![B_e]\!]_k = \begin{bmatrix} P\bar{B}_s(k) \\ \bar{B}_s(k) \\ [\![\bar{B}_p]\!]_k \end{bmatrix} \quad \text{with } P = [I_2 \ 0].$$

To see if this upper bound is overly conservative, we can calculate the ℓ_2-induced norm of the LTV system $G_\delta - G_{\delta,r}$ for some particular value of Δ_e; this constitutes a lower bound on the supremum. For instance, in this example, we find that

$$3.5 \times 10^{-5} \le \sup_\Delta \|G_\delta - G_{\delta,r}\| \le 8.2 \times 10^{-5}$$

where the lower bound corresponds to $\Delta_e = I$. It might be possible to improve on the upper bound given by the optimization problem if we allow for eventually periodic solutions of the KYP inequality with larger finite horizons, as discussed in [1]. Also, to calculate the ℓ_2-induced norm of the eventually periodic LTV system, we use the version of the KYP lemma for such systems (see [17]–[19]). Bearing in mind the simple approach we use to compute the error bound E, we find that this bound is of the same order of magnitude as the worst case error.

Error bounds are usually used as a guideline for model reduction; namely, a truncation is justified if the associated error bound, say E, is insignificant relative to the norm of G_δ for all Δ, i.e., $E \ll \inf_\Delta \|G_\delta\|$. However, computing this infimum or at least a good lower bound on its value does not appear to be easily achievable with current results. We will just touch upon this problem here and hopefully investigate it further in future work. To start, note that solving $\inf_\Delta \|G_\delta\|$ can be equivalently reformulated as a synthesis control problem of finding a controller $K = \Delta_p$ for the eventually periodic LTV system G, which would minimize the ℓ_2-induced norm of the closed-loop system. Since, in this case, the direct feedthrough term, i.e., the "D_{22}" operator, is A_{pp} and hence nonzero, then the results of [17]–[19] cannot be applied directly; however this can be remedied easily by setting $K = \Delta_p(I - A_{pp}\Delta_p)^{-1}$. The results of the preceding references minimize the closed-loop norm over dynamic controllers, and hence the solution will most likely be an overly conservative lower bound on the infimum. This bound could be improved if we restrict the search to controllers that are static but unbounded to avoid an NP-hard problem [23]. Then again, this is a problem in itself, and a good starting point would be to generalize and maybe improve on the methods of [24], [25] among others to devise a procedure for minimizing the closed-loop norm over static controllers. It is worth noting that a special case of this $\inf_\Delta \|G_\delta\|$ problem is solved in [26], where the LFT considered is an interconnection between a continuous linear time-invariant system and a diagonal matrix of design parameters. Of course, if the state space operator D of system G_δ is nonzero and invertible, then one lower bound would be

$$\inf_\Delta \|G_\delta\| \ge \frac{1}{\sup_\Delta \|G_\delta^{-1}\|} \ge \frac{1}{\alpha}$$

where α satisfies the condition $\sup_\Delta \|G_\delta^{-1}\| \le \alpha$ and is provided by applying the KYP lemma as discussed previously; this holds assuming that $\mathcal{G}_\delta^{-1}$ is strongly ℓ_2-stable. As far as this example is concerned, we have $5.1381 \times 10^{-2} \le \sup_\Delta \|G_\delta\| \le 5.1388 \times 10^{-2}$, where the lower bound corresponds to $\Delta_p = 0$; then, since we have set up the problem so that the reduction of the parameters results in a negligible error, it is reasonable to assume that the value of $\inf_\Delta \|G_\delta\|$ is close to 0.05.

Remark 22: If we only have one parameter $\bar{\delta}$, then, since the minimality theory for transfer functions in a single complex variable is identical to that for rational functions in a single real variable, it might be possible to reduce the model dimensions pointwise in time at no cost. Specifically, appealing to the state–space equations (19), we have

$$x(k+1) = \underbrace{\left(\bar{\delta}(k)I \star \left[\begin{array}{c|cc} A_{pp} & A_{ps} & B_p \\ \hline A_{sp}(k) & A_{ss}(k) & B_s(k) \end{array}\right]\right)}_{\mathcal{H}(\bar{\delta}(k))} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

and so, at each time instant k, we can reduce the dimensions of the model by eliminating any uncontrollable or unobservable states of the real-variable "transfer function" $\mathcal{H}(\bar{\delta}(k))$, and hence obtaining the minimal realization of $\mathcal{H}(\bar{\delta}(k))$.

Remark 23: Note that, in this example, we started with a continuous-time model, but then we had to discretize this model so as to be able to apply the results of this work. An interesting problem is to examine whether similar framework and results can be developed for continuous-time NSLPV systems. In fact, in [5], this balanced truncation problem is studied in continuous and discrete-time for purely time-varying systems, and the procedure and results are shown to be very similar for both cases. However, extra care is needed in the continuous case; namely, in continuous-time, we require regularity conditions on the system realization to ensure the existence of a well-behaved balancing transformation. Finding this transformation is also a much more involved computational task than it is in the discrete-time case. In [27], the balanced truncation procedure is applied to a linear time-varying approximation of a diesel exhaust catalyst model, where both continuous-time and discrete-time approaches are considered. The reduced-order model, in both cases, is obtained using certain projections rather than direct balancing. In this case study, it is shown that, while both approaches give good approximations with comparable worst-case errors, the discrete-time approach is simpler to implement and requires fewer computations.

VI. Coprime Factors Reduction for Unstable NSLPV Models

This section extends the applicability of the model reduction results in Section IV to the class of strongly stabilizable and detectable NSLPV models. In fact, by generalizing the coprime factors reduction methods given in [6]–[8], we can still utilize the balanced truncation technique of Section IV to systematically reduce strongly stabilizable and detectable NSLPV models, and further maintain some means of evaluating the error resulting from the reduction process.

A. Strongly Stabilizable and Detectable NSLPV Models

This subsection defines and further provides tools to identify strongly stabilizable and detectable NSLPV models. Note that the results here are generalizations of their counterpart matrix results given in [2]. To start, we define the set $\mathcal{F}$ to consist of all the operators $F \in \mathcal{L}_c(\ell_2^{(n_0,\ldots,n_d)}, \ell_2(\mathbb{R}^{n_w}))$ having the form $F = [F_0\ F_1\ \ldots\ F_d]$, where each $F_i \in \mathcal{L}_c(\ell_2(\mathbb{R}^{n_i}), \ell_2(\mathbb{R}^{n_w}))$ is *block-diagonal*.

Definition 24: An NSLPV model is strongly stabilizable by a feedback operator $F \in \mathcal{F}$ if: i) it is well-posed and ii) the resulting closed-loop NSLPV realization is strongly ℓ_2-stable, i.e., there exists $P \in \mathcal{X}$ such that

$$(A+BF)P(A+BF)^* - \tilde{Z}^* P \tilde{Z} \prec 0. \tag{20}$$

Similarly, an NSLPV model is strongly detectable if it is well-posed and if there exist $Q \in \mathcal{X}$ and a bounded operator L, of appropriate dimensions and similar structure to that of F^*, such that

$$(A+LC)^* \tilde{Z}^* Q \tilde{Z}(A+LC) - Q \prec 0. \tag{21}$$

To prove the next proposition, we require the following result from [14], which is based on the matrix version in [28].

Lemma 25: Given partitioned bounded operators H, U, and V, each of whose elements is a block-diagonal operator, with H being self-adjoint, then there exists a partitioned bounded operator J of compatible block-diagonal elements satisfying

$$H + U^* J V + V^* J^* U \prec 0$$

if and only if

$$W_U^* H W_U \prec 0 \quad \text{and} \quad W_V^* H W_V \prec 0 \tag{22}$$

where $\mathrm{Im}\ W_U = \mathrm{Ker}\ U, \mathrm{Im}\ W_V = \mathrm{Ker}\ V, W_U^* W_U = I$, and $W_V^* W_V = I$.

Proposition 26: The following are equivalent.

i) There exist $P \in \mathcal{X}, F \in \mathcal{F}$ such that

$$(A+BF)P(A+BF)^* - \tilde{Z}^* P \tilde{Z} \prec 0.$$

ii) There exists $Q \in \mathcal{X}$ such that $AQA^* – \tilde{Z}^* Q \tilde{Z} - BB^* \prec 0$.

Proof: For notational simplicity, we assume in this proof that the matrices $[\![B]\!]_k$ are rank-deficient for all $k \geq 0$; the proof for the general setting follows immediately. Also, we assume without loss of generality that $\mathrm{rank}[\![B]\!]_k = n_w(k) < n_0(k) + n(k)$ for all $k \geq 0$, since otherwise there are redundancies in the controls which can be easily removed. Given these assumptions, we can always find an operator $B_\perp$ of the same structure as B such that $B_\perp^* B_\perp = I, B^* B_\perp = 0$, and $[B\ \ B_\perp]$ has a bounded inverse.

Applying the Schur complement formula to the inequality in i), we get the equivalent inequality

$$\begin{bmatrix} -P^{-1} & (A+BF)^* \\ (A+BF) & -\tilde{Z}^* P \tilde{Z} \end{bmatrix} \prec 0$$

which can be equivalently written as

$$\begin{bmatrix} -P^{-1} & A^* \\ A & -\tilde{Z}^* P \tilde{Z} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} F [I \quad 0] + \begin{bmatrix} I \\ 0 \end{bmatrix} F^* [0 \quad B^*] \prec 0.$$

Setting $H = \begin{bmatrix} -P^{-1} & A^* \\ A & -\tilde{Z}^* P \tilde{Z} \end{bmatrix}, U = [0\ \ B^*], V = [I\ \ 0]$, it is obvious from Lemma 25 that i) is equivalent to (22), where $W_U = \mathrm{diag}(I, B_\perp)$ and $W_V = [0\ \ I]^*$. The condition $W_V^* H W_V \prec 0$ is trivial. Performing the multiplication $W_U^* H W_U$ and then applying the Schur complement formula to the resulting inequality lead to

$$B_\perp^* (APA^* - \tilde{Z}^* P \tilde{Z}) B_\perp \prec 0. \tag{23}$$

Last, the existence of a solution in $\mathcal{X}$ to (23) is indeed equivalent to ii); this follows from an immediate generalization of Finsler's Lemma along with scaling. ■

Theorem 27: An NSLPV model is strongly stabilizable by a feedback operator $F \in \mathcal{F}$ if and only if there exists $P \in \mathcal{X}$ such that

$$APA^* - \tilde{Z}^* P \tilde{Z} - BB^* \prec 0. \tag{24}$$

Furthermore, $F = -(B^* \tilde{Z}^* P^{-1} \tilde{Z} B)^{-1} B^* \tilde{Z}^* P^{-1} \tilde{Z} A$, if well-defined, is one such stabilizing operator.

Proof: This result follows from Definition 24 and Proposition 26. We only need to show that the above specific choice of operator F works. To start, since $P \succ 0$, then assuming that $B^* B$ is invertible in $\mathcal{L}_c(\ell_2)$ ensures that this particular value of F is well-defined; this assumption can always be achieved by removing any redundancies in the controls and slightly perturbing B if necessary. Applying the Schur complement formula twice to inequality (24), we get the equivalent inequality $-P^{-1}+A^*(\tilde{Z}^*P\tilde{Z}+BB^*)^{-1}A \prec 0$. An immediate generalization of the matrix inverse lemma gives the following:

$$(\tilde{Z}^*P\tilde{Z}+BB^*)^{-1} = \tilde{Z}^*P^{-1}\tilde{Z} - \tilde{Z}^*P^{-1}\tilde{Z}B(I+B^*\tilde{Z}^*P^{-1}\tilde{Z}B)^{-1}B^*\tilde{Z}^*P^{-1}\tilde{Z}.$$

Then it is not difficult to see from the preceding that

$$-P^{-1}+A^*\tilde{Z}^*P^{-1}\tilde{Z}A - A^*\tilde{Z}^*P^{-1}\tilde{Z}B(B^*\tilde{Z}^*P^{-1}\tilde{Z}B)^{-1}B^*\tilde{Z}^*P^{-1}\tilde{Z}A \prec 0$$

which can be equivalently written as

$$-P^{-1}+(A+BF)^*\tilde{Z}^*P^{-1}\tilde{Z}(A+BF) \prec 0$$

where $F = -(B^*\tilde{Z}^*P^{-1}\tilde{Z}B)^{-1}B^*\tilde{Z}^*P^{-1}\tilde{Z}A$. Then applying the Schur complement formula twice to the last inequality, we get inequality (20). ■

Corollary 28: An NSLPV model is strongly detectable if and only if there exists $Q \in \mathcal{X}$ such that $A^*\tilde{Z}^*Q\tilde{Z}A - Q - C^*C \prec 0$. The proof of this result parallels that of Theorem 27.

B. Right-Coprime Factorization

In this subsection, we define a notion of right-coprime factorization for NSLPV systems. Also, we show that a strongly stabilizable and detectable NSLPV system always has a right-coprime factorization.

Definition 29: Two operators N_δ and M_δ in $\mathcal{L}_c(\ell_2)$ are right-coprime if there exist U_δ, V_δ in $\mathcal{L}_c(\ell_2)$ such that

$$U_\delta N_\delta + V_\delta M_\delta = I. \tag{25}$$

Moreover, two ℓ_2-stable NSLPV systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are right-coprime if N_δ and M_δ are right-coprime for all $\Delta \in \mathbf{\Delta}$.

Definition 30: Given an NSLPV system $\mathcal{G}_\delta$, then the ℓ_2-stable NSLPV system pair $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ is a right-coprime factorization of $\mathcal{G}_\delta$ if, for all $\Delta \in \mathbf{\Delta}$, we have

a) M_δ is invertible in $\mathcal{L}_e(\ell)$;
b) N_δ and M_δ are right-coprime;
c) $G_\delta = N_\delta M_\delta^{-1}$.

Proposition 31: Given a strongly stabilizable and detectable NSLPV model $\mathcal{G}_\delta$ with a stabilizing feedback operator $F \in \mathcal{F}$, then the strongly ℓ_2-stable NSLPV system pair

$$\mathcal{N}_\delta = \mathbf{\Delta} \star \left[\begin{array}{c|c} \tilde{Z}(A+BF) & \tilde{Z}B \\ \hline C+DF & D \end{array}\right]$$
$$\mathcal{M}_\delta = \mathbf{\Delta} \star \left[\begin{array}{c|c} \tilde{Z}(A+BF) & \tilde{Z}B \\ \hline F & I \end{array}\right] \tag{26}$$

is a right-coprime factorization for $\mathcal{G}_\delta$.

Proof: In the following, fix Δ to be some element in $\mathbf{\Delta}$. To start, it is clear from (26) that N_δ and M_δ are in $\mathcal{L}_c(\ell_2)$. Referring to Definition 30, we first need to show that M_δ is invertible in $\mathcal{L}_e(\ell)$, and for that matter, consider

$$R_\delta = \Delta \star \left[\begin{array}{c|c} \tilde{Z}A & \tilde{Z}B \\ \hline -F & I \end{array}\right].$$

As G_δ is well-posed, then so is R_δ. Also, it can be easily verified that $R_\delta M_\delta = M_\delta R_\delta = I$, i.e., $R_\delta = M_\delta^{-1}$. Hence, the inverse of M_δ exists and is well-defined on ℓ. To prove $G_\delta = N_\delta M_\delta^{-1}$, first notice that

$$N_\delta M_\delta^{-1} = \begin{bmatrix} \Delta & \\ & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} \tilde{Z}(A+BF) & -\tilde{Z}BF & \tilde{Z}B \\ 0 & \tilde{Z}A & \tilde{Z}B \\ \hline C+DF & -DF & D \end{array}\right] = \bar{\Delta} \star \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where $\bar{A}, \bar{B}, \bar{C}$, and $\bar{\Delta}$ are defined in the obvious way. Applying a state space transformation to the previous realization of the system $N_\delta M_\delta^{-1}$, we get

$$N_\delta M_\delta^{-1} = (T^{-1}\bar{\Delta}T) \star \left[\begin{array}{c|c} T^{-1}\bar{A}T & T^{-1}\bar{B} \\ \hline \bar{C}T & D \end{array}\right]$$
where $T = \begin{bmatrix} J & J \\ 0 & J \end{bmatrix}$

and $J = \operatorname{diag}(I_{\ell_2}^{n_0}, I_{\ell_2}^{n_1}, \ldots, I_{\ell_2}^{n_d})$. The result is an *exactly reducible* LFT realization which can be equivalently reduced to the realization $(A, B, C, D; \Delta)$ of G_δ, namely

$$N_\delta M_\delta^{-1} = \begin{bmatrix} \Delta & \\ & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} \tilde{Z}(A+BF) & 0 & 0 \\ 0 & \tilde{Z}A & \tilde{Z}B \\ \hline C+DF & C & D \end{array}\right] = \Delta \star \left[\begin{array}{c|c} \tilde{Z}A & \tilde{Z}B \\ \hline C & D \end{array}\right] = G_\delta.$$

Last, we show that N_δ and M_δ are right-coprime. Since $\mathcal{G}_\delta$ is strongly detectable, then, by Definition 24, there exists a bounded operator L such that $A+LC$ is a strongly stable operator. Then, following a similar argument as before, it is not difficult to verify that U_δ and V_δ, defined later, satisfy condition (25)

$$U_\delta = \Delta \star \left[\begin{array}{c|c} \tilde{Z}(A+LC) & \tilde{Z}L \\ \hline F & 0 \end{array}\right] \in \mathcal{L}_c(\ell_2)$$
$$V_\delta = \Delta \star \left[\begin{array}{c|c} \tilde{Z}(A+LC) & \tilde{Z}(B+LD) \\ \hline -F & I \end{array}\right] \in \mathcal{L}_c(\ell_2).$$

C. Coprime Factors Reduction

This subsection presents a systematic approach for the model reduction of strongly stabilizable and detectable NSLPV systems. This approach is a generalization of the coprime factors reduction method first proposed by Meyer in [8] for standard state-space systems, and later generalized to the class of stationary LPV and uncertain systems in [6], [7]. The error measure here, while still norm-based, does not directly capture the mismatch between the nominal system and the reduced-order model, as is the case in Section IV where strongly ℓ_2-stable models are considered. Instead, the measure we use in this case is related to the closed-loop stability of these two systems. Hence, given a nominal NSLPV plant $\mathcal{G}_\delta$ and a corresponding right-coprime factorization $(\mathcal{N}_\delta, \mathcal{M}_\delta)$, defined as in (26), we construct the strongly ℓ_2-stable NSLPV model

$$\mathcal{H}_\delta = \begin{bmatrix} \mathcal{N}_\delta \\ \mathcal{M}_\delta \end{bmatrix} = \mathbf{\Delta} \star \left[\begin{array}{c|c} \tilde{Z}(A+BF) & \tilde{Z}B \\ \hline C+DF & D \\ F & I \end{array}\right]. \tag{27}$$

Using model reduction for strongly ℓ_2-stable systems, as discussed in Section IV, we find an approximation using the error measure

$$\sup_{\Delta\in\mathbf{\Delta}}\left\|\begin{bmatrix}N_\delta\\M_\delta\end{bmatrix}-\begin{bmatrix}N_{\delta,r}\\M_{\delta,r}\end{bmatrix}\right\|.$$

We can then directly relate the factorization $(\mathcal{N}_{\delta,r},\mathcal{M}_{\delta,r})$ to a lower order NSLPV system $\mathcal{G}_{\delta,r}$, namely $G_{\delta,r}=N_{\delta,r}M_{\delta,r}^{-1}$.

The following is an outline of the coprime factors reduction algorithm for a strongly stabilizable and detectable NSLPV model:

$$\mathcal{G}_\delta=\mathbf{\Delta}\star\left[\begin{array}{c|c}\tilde{Z}A & \tilde{Z}B\\ \hline C & D\end{array}\right].$$

1) Find $P\in\mathcal{X}$ satisfying inequality (24), then set $F=-(B^*\tilde{Z}^*P^{-1}\tilde{Z}B)^{-1}B^*\tilde{Z}^*P^{-1}\tilde{Z}A$; this choice of a stabilizing feedback operator for $\mathcal{G}_\delta$ can always be achieved by removing any redundancies in the controls, so that $[\![B]\!]_k$ has full-column rank for all $k\geq 0$, and by slightly perturbing B if necessary to ensure boundedness.
2) Construct a right-coprime factorization for $\mathcal{G}_\delta$, for instance the one given in (26), and a corresponding strongly ℓ_2-stable NSLPV model $\mathcal{H}_\delta$, defined as in (27).
3) Find $X,Y\in\mathcal{X}$ satisfying the generalized Lyapunov inequalities for $\mathcal{H}_\delta$, namely

$$(A+BF)X(A+BF)^*-\tilde{Z}^*X\tilde{Z}+BB^*\prec 0 \quad (28)$$
$$(A+BF)^*\tilde{Z}^*Y\tilde{Z}(A+BF)-Y+(C+DF)^*(C+DF)+F^*F\prec 0. \quad (29)$$

We usually require X,Y to satisfy some criterion that yields reasonable error bounds, for example, selecting the solution with the minimum trace. Note that X (or Y) might also have to satisfy an additional condition to ensure the well-posedness of the reduced model $\mathcal{G}_{\delta,r}$, as we will discuss later in this subsection.
4) Use X and Y to construct a balancing transformation $T\in\mathcal{T}$ and a diagonal gramian $\Sigma\in\mathcal{X}$, as outlined for the example in Section V, where Σ satisfies the Lyapunov inequalities for the following balanced system realization of

$$\mathcal{H}_\delta:\ \mathcal{H}_\delta=\mathbf{\Delta}\star\left[\begin{array}{c|c}\tilde{Z}((\tilde{Z}^*T\tilde{Z})(A+BF)T^{-1}) & \tilde{Z}((\tilde{Z}^*T\tilde{Z})B)\\ \hline (C+DF)T^{-1} & D\\ FT^{-1} & I\end{array}\right].$$

5) Invoke Theorems 13, 15, and 17 to compute error bounds from the gramian Σ, which would serve as a guideline for the balanced truncation. Say, one convenient reduced model is $\mathcal{H}_{\delta,r}$ with reduced dimensions r_i, where we have $0\leq r_i(k)\leq n_i(k)$ for all $i=0,1,\ldots,d$ and $k\geq 0$. Define the truncation operator $\mathcal{P}=\mathrm{diag}(\mathcal{P}_0,\mathcal{P}_1,\ldots,\mathcal{P}_d)$, where each $\mathcal{P}_i$ is a block-diagonal operator composed of matrices $\mathcal{P}_i(k)=[I_{r_i(k)}\ 0]\in\mathbb{R}^{r_i(k)\times n_i(k)}$. Then, the truncated system $\mathcal{H}_{\delta,r}$ is given by

$$\mathcal{H}_{\delta,r}=\mathbf{\Delta}_r\star\left[\begin{array}{c|c}\tilde{Z}((\tilde{Z}^*\mathcal{P}T\tilde{Z})(A+BF)T^{-1}\mathcal{P}^*) & \tilde{Z}((\tilde{Z}^*\mathcal{P}T\tilde{Z})B)\\ \hline (C+DF)T^{-1}\mathcal{P}^* & D\\ FT^{-1}\mathcal{P}^* & I\end{array}\right]$$

where $\mathbf{\Delta}_r=\{\mathcal{P}\Delta\mathcal{P}^*:\Delta\in\mathbf{\Delta}\}$.
6) Set $A_r=(\tilde{Z}^*\mathcal{P}T\tilde{Z})AT^{-1}\mathcal{P}^*, B_r=(\tilde{Z}^*\mathcal{P}T\tilde{Z})B, C_r=CT^{-1}\mathcal{P}^*$, and $F_r=FT^{-1}\mathcal{P}^*$. Then, the strongly ℓ_2-stable balanced truncation $\mathcal{H}_{\delta,r}$ provides the right-coprime NSLPV systems

$$\mathcal{N}_{\delta,r}=\mathbf{\Delta}_r\star\left[\begin{array}{c|c}\tilde{Z}(A_r+B_rF_r) & \tilde{Z}B_r\\ \hline C_r+DF_r & D\end{array}\right]$$
$$\mathcal{M}_{\delta,r}=\mathbf{\Delta}_r\star\left[\begin{array}{c|c}\tilde{Z}(A_r+B_rF_r) & \tilde{Z}B_r\\ \hline F_r & I\end{array}\right].$$

Assuming that $I-\Delta_r\tilde{Z}A_r$ *is invertible in* $\mathcal{L}_e(\ell)$ *for all* $\Delta_r\in\mathbf{\Delta}_r$, then the system pair $(\mathcal{N}_{\delta,r},\mathcal{M}_{\delta,r})$ constitutes a right-coprime factorization for the reduced NSLPV model $\mathcal{G}_{\delta,r}$, which is given by

$$\mathcal{G}_{\delta,r}=\mathbf{\Delta}_r\star\left[\begin{array}{c|c}\tilde{Z}A_r & \tilde{Z}B_r\\ \hline C_r & D\end{array}\right].$$

Note that, in this case, F_r is a stabilizing feedback operator for $\mathcal{G}_{\delta,r}$.

We now briefly comment on the previous algorithm. For simplicity, given $Q=\mathrm{diag}(Q_0,Q_1,\ldots,Q_d)$, we define the notation $Q_p=\mathrm{diag}(Q_1,Q_2,\ldots,Q_d)$. The reduced NSLPV model $\mathcal{G}_{\delta,r}$ may not be well-posed in general, and hence the assumption in step 6), which is equivalent to saying that $I-\Delta_{p,r}A_{pp,r}$ is invertible in $\mathcal{L}_e(\ell)$ for all $\Delta_r=\mathrm{diag}(I_{\ell_2}^{r_0},\Delta_{p,r})\in\mathbf{\Delta}_r$, where $A_{pp,r}=\mathcal{P}_pT_pA_{pp}T_p^{-1}\mathcal{P}_p^*$ and $\Delta_{p,r}=\mathcal{P}_p\Delta_p\mathcal{P}_p^*$. The well-posedness of the approximation $\mathcal{G}_{\delta,r}$ can be guaranteed if the gramian X satisfying (28) further satisfies the condition $A_{pp}X_pA_{pp}^*-X_p\prec 0$. This can be easily shown as follows: recalling from step 4) that $TXT^*=\Sigma$, where Σ is partitioned as in (10), then it follows that $A_{pp,r}\Gamma_pA_{pp,r}^*-\Gamma_p\prec 0$, which in turn implies that $I-\Delta_{p,r}A_{pp,r}$ is invertible in $\mathcal{L}_c(\ell_2)$ for all $\Delta_{p,r}$. Alternatively, we can impose a similar condition on the observability gramian Y to ensure well-posedness.

Next, we give two results pertaining to eventually periodic models.

Theorem 32: Suppose that A and B are (h,q)-eventually periodic. Then there exists an operator $P\in\mathcal{X}$ solving inequality (24) if and only if there exists an (h,q)-eventually periodic solution $P_{\mathrm{eper}}\in\mathcal{X}$.

The proof is similar to that of Theorem 19. The next result stems from Theorems 27 and 32.

Corollary 33: Given a strongly stabilizable (h,q)-eventually periodic LPV model, then one (h,q)-eventually periodic stabilizing feedback operator $F_{\mathrm{eper}}\in\mathcal{F}$ is given by

$$F_{\mathrm{eper}}=-\left(B^*\tilde{Z}^*P_{\mathrm{eper}}^{-1}\tilde{Z}B\right)^{-1}B^*\tilde{Z}^*P_{\mathrm{eper}}^{-1}\tilde{Z}A \quad (30)$$

where P_{eper} is an (h,q)-eventually periodic operator in $\mathcal{X}$ satisfying inequality (24).

Thus, in the case of (h,q) eventually periodic LPV plants, all the NSLPV models encountered in the preceding model reduction algorithm can be chosen to be (h,q)-eventually periodic.

Remark 34: The error bounds we get here are given in terms of the distance between the coprime factors realizations, and thus have an interpretation in terms of robust feedback stability. Connections can also be made to the graph or gap metric settings with connotations for control design robustness analysis, or more generally for robustness of interconnections; however, the mathematical details are still lacking. These robustness implications are briefly addressed for the stationary LPV case in [7, Sec. 4]. Also, as stated for the stationary case in [7], while the coprime factors are usually *normalized* in the standard state-space case to ensure the least conservative robustness conditions, constructing normalized coprime factors in our case is certainly a formidable task due to the spatial structure limitations. Instead, *contractive* and *expansive* coprime factors realizations may be considered; see [7] and [29] for more on this.

VII. Conclusion

In this paper, we have introduced balanced truncation model reduction for NSLPV systems, derived explicit error bounds for this procedure, and applied these results to an eventually periodic translational system. Even when restricted to purely time-varying systems, the results obtained provide the least conservative bounds currently available in the literature. Although there has been considerable recent achievement in the literature on model reduction of nonstationary systems, which are all directly motivated by the original LTI results in [12], [13], we conjecture that significantly better bounds may be obtainable. Furthermore, we have extended the coprime factors reduction approach to the class of NSLPV systems. The motivation for this work is that NSLPV models used to represent even low-dimensional nonlinear systems around trajectories can be of very high order. The results of this paper can be used to predictably reduce such models to manageable complexity levels in various design scenarios, including distributed or networked situations where the topological structure is to be preserved.

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