On the Balanced Truncation of LTV Systems

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***Abstract*—This note furthers results on the balanced truncation of stable linear time-varying discrete-time systems. The main result contributes better error bounds than the currently available ones in certain model reduction scenarios; this is demonstrated in an example of a four-mass translational system. Also, this note gives new finite error bounds for the balanced truncation of stable eventually periodic models.**

***Index Terms*—Balanced truncation, model reduction, time-varying systems.**

I. Introduction

This note provides new error bounds for the balanced truncation model reduction of stable linear time-varying (LTV) discrete-time systems. The main theorem complements recent results on the model reduction of LTV systems, notably those of [1], [2], providing tighter error bounds in certain scenarios. Also, this note gives new finite error bounds for the balanced truncation of stable *eventually periodic* systems; such systems are aperiodic for an initial amount of time.

Our treatment of balanced truncation model reduction follows that of [1], where the notion of balanced realization is defined in terms of generalized gramians, which are solutions of strict Lyapunov inequalities. The approach here, like in [1], utilizes the operator theoretic framework developed in [3] for analysis of LTV systems. Given a stable discrete-time LTV system G with time-varying state dimension n_k where k is time, we would like to apply the balanced truncation model reduction procedure to obtain a reduced-order model G_r of dimension $r_k \ll n_k$ for all k in some time interval T, such that the error measure, defined in terms of the ℓ_2-induced norm $\|G - G_r\|$, is smaller than some chosen threshold. For that matter, it is very useful to have explicit error bounds, given in terms of the singular values of the generalized gramians, that would serve as a guideline for obtaining a reduced model that properly represents the original system. The first such error bounds are given in [1] and [4], but they can be quite conservative unless the singular values of the gramians are equal over time. These bounds are significantly improved in [2], specifically in cases where the singular values vary monotonically over time, or if nonmonotonically, then relatively slowly; in the event of rapid variations, unnecessarily conservative bounds may be avoided by splitting the time interval over which the truncation is implemented into a number, say m, of intervals, and then truncating the states in m steps. Concerning the result of this note, if we are to truncate the last $n_k - r_k$ states over some time intervals, then we only need to account for the singular values corresponding to the (r_k+1)-th state when computing the error bound. Hence, our result has a clear advantage over those of [1] and [2] when truncating lots of states, i.e., $r_k \ll n_k$, over small time intervals, where the error bounds given by [1] and [2] for truncating the last $n_k - r_k + 1$ states are relatively significant. The bounds given here can also be quite handy when truncating eventually periodic systems exhibiting short transient time variations, as shown in the example at the end of the note.

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II. Preliminaries

The set of real numbers and that of real $n \times m$ matrices are denoted by $\mathbb{R}$ and $\mathbb{R}^{n\times m}$, respectively. The maximum singular value of a matrix X is denoted by $\bar{\sigma}(X)$.

Given two Hilbert spaces E and F, we denote the space of bounded linear operators mapping E to F by $\mathcal{L}(E, F)$, and shorten this to $\mathcal{L}(E)$ when E equals F. If X is in $\mathcal{L}(E, F)$, we denote the E to F induced norm of X by $\|X\|_{E\to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. When an operator $X \in \mathcal{L}(E)$ is self-adjoint, we use $X \prec 0$ to mean it is negative definite; that is there exists a number $\alpha > 0$ such that, for all nonzero $x \in E$, the inequality $\langle x, Xx\rangle < -\alpha\|x\|^2$ holds, where $\langle\cdot,\cdot\rangle$ denotes the inner product and $\|\cdot\|$ denotes the corresponding norm on E. If S_i is a sequence of operators, then $\mathrm{diag}(S_i)$ denotes their block-diagonal augmentation.

The main Hilbert space of interest in this note is denoted by $\ell_2(J)$ where J is an infinite sequence of Euclidean spaces, namely $J = (\mathbb{R}^{n_0}, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \ldots)$. It consists of elements $x = (x_0, x_1, x_2, \ldots)$, with each $x_k \in \mathbb{R}^{n_k}$, which have a finite two-norm $\|x\|$ defined by $\|x\|^2 = \sum_{k=0}^{\infty}\|x_k\|^2 < \infty$. The inner product of x, y in $\ell_2(J)$ is hence defined as the sum $\langle x, y\rangle = \sum_{k=0}^{\infty}\langle x_k, y_k\rangle$. If the sequence of spaces J is clear from the context, then the notation $\ell_2(J)$ is abbreviated to ℓ_2. Given a time-varying dimension n_k, we define the notation $I_{\ell_2}^n := \mathrm{diag}(I_{n_0}, I_{n_1}, I_{n_2}, \ldots)$, where I_{n_i} is an $n_i \times n_i$ identity matrix.

A key operator used in this note is the unilateral shift Z, defined as follows:

$$Z : \ell_2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \ldots) \to \ell_2(\mathbb{R}^{n_0}, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \ldots)$$
$$(a_1, a_2, \ldots) \overset{Z}{\longmapsto} (0, a_1, a_2, \ldots).$$

Following the notation and approach in [3], we make the following definitions. First, we say a bounded linear operator Q mapping $\ell_2(\mathbb{R}^{m_0}, \mathbb{R}^{m_1}, \ldots)$ to $\ell_2(\mathbb{R}^{n_0}, \mathbb{R}^{n_1}, \ldots)$ is *block-diagonal* if there exists a sequence of matrices Q_k in $\mathbb{R}^{n_k\times m_k}$ such that, for all w, z, if $z = Qw$, then $z_k = Q_k w_k$. Then Q has the representation $\mathrm{diag}(Q_0, Q_1, Q_2, \ldots)$. A *diagonal* operator is a block-diagonal operator where each of the matrix blocks is diagonal.

Suppose F, G, R, and S are block-diagonal operators, and let A be a *partitioned* operator of the form

$$A = \begin{bmatrix} F & G \\ R & S \end{bmatrix}.$$

Then, we define the following notation:

$$[\![A]\!] := \mathrm{diag}\left(\begin{bmatrix} F_0 & G_0 \\ R_0 & S_0 \end{bmatrix}, \begin{bmatrix} F_1 & G_1 \\ R_1 & S_1 \end{bmatrix}, \ldots\right)$$

which we call the *diagonal realization* of A. Clearly for any given operator A of this particular structure, $[\![A]\!]$ is simply A with the rows and columns permuted appropriately so that

$$[\![A]\!]_k = \begin{bmatrix} F_k & G_k \\ R_k & S_k \end{bmatrix}.$$

From this definition, it is easy to see that $[\![A + B]\!] = [\![A]\!] + [\![B]\!]$ and $[\![AC]\!] = [\![A]\!]\,[\![C]\!]$ hold for appropriately dimensioned operators, and similarly that $A \prec \beta I$ holds if and only if $[\![A]\!] \prec \beta I$, where β is a scalar. Namely, the $[\![\bullet]\!]$ operation is a homomorphism from partitioned operators with block-diagonal entries to block-diagonal operators.

III. LTV Systems and Balanced Truncation

We now review the balanced truncation method of model reduction for discrete-time LTV state space systems. As our presentation is rather brief, we refer the reader to [1] for an in-depth treatment of the theory. To start, consider the following time-varying state space equations:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, \qquad x_0 = 0 \\ y_k &= C_k x_k + D_k u_k \end{aligned}$$

where $A_k \in \mathbb{R}^{n_{k+1} \times n_k}$, $B_k \in \mathbb{R}^{n_{k+1} \times n_{uk}}$, $C_k \in \mathbb{R}^{n_{yk} \times n_k}$, and $D_k \in \mathbb{R}^{n_{yk} \times n_{uk}}$ are bounded *real* matrix sequences. Clearly, these sequences define block-diagonal operators A, B, C and D and, therefore, the previous system may be written more compactly in operator form as

$$\begin{aligned} x &= ZAx + ZBu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where Z is the shift, or delay, operator on ℓ_2. Thus, assuming the relevant inverse exists, we can write the map from u to y as

$$u \mapsto y = C(I - ZA)^{-1} ZB + D.$$

We will say that the previous LTV state–space system is *stable* when $I - ZA$ has a bounded inverse; this is equivalent to exponential stability as shown in [1] and [3].

At this point, we define $\mathcal{X}$ as the set of positive definite block-diagonal operators $X \in \mathcal{L}(\ell_2)$ of the form $X = \operatorname{diag}(X_0, X_1, \ldots)$, where $X_i \in \mathbb{R}^{n_i \times n_i}$. We now give an explicit definition of a *balanced* realization for LTV systems.

Definition 1: An LTV realization (A, B, C, D) is balanced if there exists a *diagonal* operator $\Sigma \in \mathcal{X}$ satisfying both of the following generalized Lyapunov inequalities:

$$A\Sigma A^* - Z^*\Sigma Z + BB^* \prec 0 \tag{2}$$

$$A^* Z^* \Sigma Z A - \Sigma + C^* C \prec 0. \tag{3}$$

Thus, operator Σ is of the form $\Sigma = \operatorname{diag}(\Sigma_0, \Sigma_1, \Sigma_2, \ldots)$, where each matrix Σ_i is diagonal and belongs to $\mathbb{R}^{n_i \times n_i}$. We assume throughout without loss of generality that, in each block Σ_i, the diagonal entries are ordered with the largest first. Now given the integers r_i such that $0 \le r_i \le n_i$ for all $i \ge 0$, we partition each of the Σ_i blocks into two sub-blocks $\Gamma_i \in \mathbb{R}^{r_i \times r_i}$ and $\Omega_i \in \mathbb{R}^{(n_i - r_i) \times (n_i - r_i)}$ so that

$$\Sigma = \left[\!\!\left[\begin{matrix} \Gamma & \\ & \Omega \end{matrix} \right]\!\!\right] \tag{4}$$

where Γ and Ω are block-diagonal operators. The singular values corresponding to the states that will be truncated are in Ω. Note that if for some i we have $n_i = r_i$, then the dimension of Ω_i is zero and no states are truncated at the ith point in time.

Partitioning A, B and C conformably with the partitioning of Σ, we get

$$A = \left[\!\!\left[\begin{matrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{matrix} \right]\!\!\right] \quad B = \left[\!\!\left[\begin{matrix} \hat{B}_1 \\ \hat{B}_2 \end{matrix} \right]\!\!\right] \quad C = [\![\hat{C}_1 \quad \hat{C}_2]\!].$$

Then, the state–space realization for the balanced truncation G_r of system G is $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, D)$. By [1, Lemma 14], this reduced realization is also stable and balanced. General upper bounds on the error induced in such a reduction process are given in [1] and [2], with [2, Th. 2] providing to our knowledge the tightest currently available.

IV. Main Result

We start this section with the following result from [3].

Lemma 2: Given an LTV system realization (A, B, C, D) and an integer $\gamma > 0$, then the system is stable and satisfies the norm condition $\|C(I - ZA)^{-1} ZB + D\| < \gamma$ if and only if there exists $X \in \mathcal{X}$ satisfying

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} Z^* X Z & \\ & \frac{1}{\gamma^2} I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & \\ & I \end{bmatrix} \prec 0. \tag{5}$$

To best present our result, we will make the assumption that there are a *finite* number of matrices Ω_i with nonzero dimensions; that is, we only intend at this point to remove states at a finite number of time instants. With this said, define the finite set $\mathcal{F} = \{k \ge 1 : r_k \ne n_k\}$. Note that $k = 0$ is always excluded from the set $\mathcal{F}$ because we assume the initial state $x_0 = 0$, which guarantees a zero error when truncating all states at time $k = 0$.

Theorem 3: Suppose that (A, B, C, D) is a balanced realization for the stable system G, and that the diagonal generalized gramian $\Sigma \in \mathcal{X}$, satisfying both of inequalities (2) and (3), is partitioned as in (4). If $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_s$ are nonempty disjoint sets whose union is $\mathcal{F}$, then the balanced truncation G_r of G satisfies the following inequality:

$$\|G - G_r\| < \sum_{i=1}^{s} \left\{ \left(\sqrt{2}\right)^{|\mathcal{F}_i|} \max_{j \in \mathcal{F}_i} \bar{\sigma}\left(\Omega_j\right) \right\} \tag{6}$$

where $|\mathcal{F}_i|$ denotes the number of elements in $\mathcal{F}_i$.

Note that, in the previous theorem, one can judiciously choose the sets $\mathcal{F}_i$ so as to minimize the right-hand side of inequality (6). The following proof is inspired by that of [1, Th. 17].

Proof: As G and G_r are both stable, then so is $G - G_r$. One realization of system $G - G_r$ is $(\bar{A}, \bar{B}, \bar{C}, 0)$, where

$$\bar{A} = \left[\!\!\left[\begin{matrix} \hat{A}_{11} & 0 \\ 0 & A \end{matrix} \right]\!\!\right] \quad \bar{B} = \left[\!\!\left[\begin{matrix} \hat{B}_1 \\ B \end{matrix} \right]\!\!\right] \quad \bar{C} = [\![-\hat{C}_1 \quad C]\!].$$

In the following, we will construct a positive–definite, block-diagonal operator X satisfying inequality (5) for the realization $(\bar{A}, \bar{B}, \bar{C}, 0)$, such that γ is equal to the right-hand side of (6). Then, invoking Lemma 2 completes the proof.

To start, we assume without any loss of generality that $\max_{i \in \mathcal{F}} \bar{\sigma}(\Omega_i) = 1$, and hence $\Omega \preceq I$; this can always be achieved by scaling inequalities (2) and (3). We will prove the claim for the case $s = 1$; the general case follows simply by the standard use of the telescoping series and triangle inequality. Next, we construct the aforesaid operator X.

Given that the diagonal operator $\Sigma \in \mathcal{X}$ satisfies inequalities (2) and (3), then direct applications of the Schur complement formula guarantee the validity of the following condition:

$$\begin{bmatrix} -R_1 & K^* \\ K & -Z^* R_2 Z \end{bmatrix} \prec 0$$

$$\text{where} \quad R_i = \left[\!\!\left[\begin{matrix} \Gamma^{-1} & 0 & 0 & 0 & 0 \\ 0 & \Omega^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_{\ell_2}^{q_i} & 0 & 0 \\ 0 & 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & 0 & \Omega \end{matrix} \right]\!\!\right]$$

$$K = \left[\!\!\left[\begin{matrix} 0 & 0 & 0 & \hat{A}_{11} & \hat{A}_{12} \\ 0 & 0 & 0 & \hat{A}_{21} & \hat{A}_{22} \\ 0 & 0 & 0 & \hat{C}_1 & \hat{C}_2 \\ \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 & 0 & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 & 0 & 0 \end{matrix} \right]\!\!\right]$$

and $q_1 = n_u, q_2 = n_y$. Define the invertible operators L and P by

$$L = \begin{bmatrix} 0 & 0 & 0 & I_{\ell_2}^r & 0 \\ I_{\ell_2}^r & 0 & 0 & I_{\ell_2}^r & 0 \\ 0 & I_{\ell_2}^{n-r} & 0 & 0 & I_{\ell_2}^{n-r} \\ 0 & 0 & I_{\ell_2}^{n_y} & 0 & 0 \\ 0 & I_{\ell_2}^{n-r} & 0 & 0 & -I_{\ell_2}^{n-r} \end{bmatrix}$$

$$P = \begin{bmatrix} I_{\ell_2}^r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\ell_2}^{n-r} \\ 0 & 0 & 0 & I_{\ell_2}^{n_u} & 0 \\ -I_{\ell_2}^r & I_{\ell_2}^r & 0 & 0 & 0 \\ 0 & 0 & I_{\ell_2}^{n-r} & 0 & 0 \end{bmatrix}.$$

Pre- and postmultiplying the previous condition by $\operatorname{diag}(P^*, L)$ and $\operatorname{diag}(P, L^*)$ respectively give the following equivalent inequality:

$$\begin{bmatrix} -P^* R_1 P & P^* K^* L^* \\ LKP & -Z^* L R_2 L^* Z \end{bmatrix} \prec 0. \quad (7)$$

Performing the multiplications in this inequality leads to

$$P^* R_1 P = \begin{bmatrix} \Gamma^{-1} + \Gamma & -\Gamma & 0 & 0 & 0 \\ -\Gamma & \Gamma & 0 & 0 & 0 \\ 0 & 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & \Omega^{-1} \end{bmatrix}$$

$$L R_2 L^* = \begin{bmatrix} \Gamma & \Gamma & 0 & 0 & 0 \\ \Gamma & \Gamma^{-1} + \Gamma & 0 & 0 & 0 \\ 0 & 0 & \Omega^{-1} + \Omega & 0 & \Omega^{-1} - \Omega \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & \Omega^{-1} - \Omega & 0 & \Omega^{-1} + \Omega \end{bmatrix}$$

$$LKP = \begin{bmatrix} M & N_{12} \\ N_{21} & -\hat{A}_{22} \end{bmatrix}$$

where $N_{21} = [\![-2\hat{A}_{21} \quad \hat{A}_{21} \quad \hat{A}_{22} \quad -\hat{B}_2]\!]$

$$M = \begin{bmatrix} \hat{A}_{11} & 0 & 0 & \hat{B}_1 \\ 0 & \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ 0 & \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ -\hat{C}_1 & \hat{C}_1 & \hat{C}_2 & 0 \end{bmatrix} \quad N_{12} = \begin{bmatrix} \hat{A}_{12} \\ \hat{A}_{12} \\ \hat{A}_{22} \\ 0 \end{bmatrix}.$$

Note that, in the preceding expressions, some of the operators might contain at certain time-instants matrices of zero dimensions. In such scenarios, the rows and columns of which the said matrices are elements would not be present, and the corresponding operator inequalities remain valid. Inequality (7), together with an application of the Schur complement formula, implies that

$$M^* \begin{bmatrix} Z^* \begin{bmatrix} \Gamma^{-1} + \Gamma & -\Gamma & 0 \\ -\Gamma & \Gamma & 0 \\ 0 & 0 & (\Omega^{-1} + \Omega)^{-1} \end{bmatrix} Z & \\ & I \end{bmatrix} M - \begin{bmatrix} \begin{bmatrix} \Gamma^{-1} + \Gamma & -\Gamma & 0 \\ -\Gamma & \Gamma & 0 \\ 0 & 0 & \Omega \end{bmatrix} & \\ & I \end{bmatrix} \prec 0. \quad (8)$$

By assumption $\Omega \preceq I$, and so $I + \Omega^2 \preceq 2I$. Now $\left(\Omega^{-1} + \Omega\right)^{-1} = \Omega^{1/2}\left(I + \Omega^2\right)^{-1}\Omega^{1/2}$, and it therefore follows that $\left(\Omega^{-1} + \Omega\right)^{-1} \succeq (1/2)\Omega$.

Setting $W = \begin{bmatrix} \Gamma^{-1} + \Gamma & -\Gamma \\ -\Gamma & \Gamma \end{bmatrix}$, the following stems from the last inequality and (8):

$$M^* \begin{bmatrix} Z^* \begin{bmatrix} W & 0 \\ 0 & \frac{1}{2}\Omega \end{bmatrix} Z & \\ & I \end{bmatrix} M - \begin{bmatrix} \begin{bmatrix} W & 0 \\ 0 & \Omega \end{bmatrix} & \\ & I \end{bmatrix} \prec 0. \quad (9)$$

Let t_i, N_i, and m be the positive integers satisfying $t_{i+1} > t_i + N_i$ for all $i = 1, 2, \ldots, m$ such that

$$\mathcal{F}_1 = \cup_{i=1}^m \{t_i, t_i+1, \ldots, t_i+N_i-1\}.$$

Define the infinite sequence $(\lambda_0, \lambda_1, \lambda_2, \ldots)$ as

$$\lambda_k = \begin{cases} 1, & \text{for } 0 \le k \le t_1 \\ (\frac{1}{2})^{k-t_1}, & \text{for } t_1+1 \le k \le t_1+N_1 \\ (\frac{1}{2})^{N_1}, & \text{for } t_1+N_1+1 \le k \le t_2 \\ (\frac{1}{2})^{N_1+k-t_2}, & \text{for } t_2+1 \le k \le t_2+N_2 \\ \vdots & \vdots \\ (\frac{1}{2})^{(|\mathcal{F}_1|-N_m)+k-t_m}, & \text{for } t_m+1 \le k \le t_m+N_m \\ (\frac{1}{2})^{|\mathcal{F}_1|}, & \text{for } k \ge t_m+N_m+1 \end{cases}$$

where $|\mathcal{F}_1| = \sum_{i=1}^m N_i$. The following ensues from inequality (9) and scaling for all $k \ge 0$:

$$M_k^* \begin{bmatrix} \begin{bmatrix} \lambda_{k+1} W_{k+1} & 0 \\ 0 & \frac{1}{2}\lambda_{k+1}\Omega_{k+1} \end{bmatrix} & \\ & \lambda_{k+1} I \end{bmatrix} M_k - \begin{bmatrix} \begin{bmatrix} \lambda_{k+1} W_k & 0 \\ 0 & \lambda_{k+1}\Omega_k \end{bmatrix} & \\ & \lambda_{k+1} I \end{bmatrix} \prec -\beta I \quad (10)$$

where β is some positive integer. Note that if we are not truncating any states at some time instant k, then the corresponding matrix Ω_k will be of zero dimensions, and so the rows and columns containing this matrix will not be present in the previous inequality. Now, the sequence λ_i is a nonincreasing sequence, and hence $(1/2)^{|\mathcal{F}_1|} \le \lambda_{k+1} \le \lambda_k \le 1$. Also, for $k \in \mathcal{F}_1$, we have $\lambda_{k+1} = (1/2)\lambda_k$. Then, the following inequality clearly holds:

$$\begin{bmatrix} \begin{bmatrix} \lambda_{k+1} W_k & 0 \\ 0 & \lambda_{k+1}\Omega_k \end{bmatrix} & \\ & \lambda_{k+1} I \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} \lambda_k W_k & 0 \\ 0 & \frac{1}{2}\lambda_k\Omega_k \end{bmatrix} & \\ & I \end{bmatrix} \preceq 0. \quad (11)$$

Adding inequalities (10) and (11), and since $\lambda_{k+1} \ge (1/2)^{|\mathcal{F}_1|}$, we obtain

$$M_k^* \begin{bmatrix} \begin{bmatrix} \lambda_{k+1} W_{k+1} & 0 \\ 0 & \frac{1}{2}\lambda_{k+1}\Omega_{k+1} \end{bmatrix} & \\ & (\frac{1}{2})^{|\mathcal{F}_1|} I \end{bmatrix} M_k - \begin{bmatrix} \begin{bmatrix} \lambda_k W_k & 0 \\ 0 & \frac{1}{2}\lambda_k\Omega_k \end{bmatrix} & \\ & I \end{bmatrix} \prec -\beta I.$$

Setting $X = \operatorname{diag}(X_0, X_1, X_2, \ldots)$, where, for all $k \ge 0$, $X_k = \operatorname{diag}(\lambda_k W_k, (1/2)\lambda_k \Omega_k) \succ 0$, then clearly the positive definite operator X solves inequality (5) for the realization $(\bar{A}, \bar{B}, \bar{C}, 0)$ of system $G - G_r$, with $\gamma = \sqrt{2}^{|\mathcal{F}_1|}$. Finally, invoking Lemma 2 and recalling that by assumption $\max \bar{\sigma}(\Omega_i) = 1$, we conclude that system $G - G_r$ satisfies the norm condition $\|G - G_r\| < \sqrt{2}^{|\mathcal{F}_1|} \max_{i \in \mathcal{F}_1} \bar{\sigma}(\Omega_i)$. ■

Remark 4: Theorem 3 immediately generalizes to the case of infinite set $\mathcal{F}$; however, in instances where the summation in (6) diverges, one cannot conclude an error bound from this result.

In Section VI, we will demonstrate the usefulness of this result via a complete example. However, for now, to illustrate how to apply Theorem 3, we consider a hypothetical example where we are to truncate the states corresponding to the following sequence $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5\}$:

$$\left\{\begin{bmatrix}5 & & \\ & 3 & \\ & & 1\end{bmatrix}, \begin{bmatrix}3.5 & & \\ & 1.5 & \\ & & 0.25\end{bmatrix}, \begin{bmatrix}4.5 & & \\ & 2.5 & \\ & & 0.75\end{bmatrix}, \begin{bmatrix}3.25 & & \\ & 1.25 & \\ & & 0.5\end{bmatrix}, \begin{bmatrix}4.75 & & \\ & 3.75 & \\ & & 1.75\end{bmatrix}\right\}.$$

Then, the corresponding error bound derived by applying effectively [2, Th. 2] is around 40, whereas if we employ Theorem 3, and choose $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2\}$, where $\mathcal{F}_1 = \{1, 3, 5\}$ and $\mathcal{F}_2 = \{2, 4\}$, then the error bound would be $\left(\sqrt{2}^3\right)(5) + \left(\sqrt{2}^2\right)(3.5) \simeq 21$, which is roughly half the value of the error bound given by the result from [2].

V. Eventually Periodic Systems

This section focuses on the balanced truncation of stable *eventually periodic* systems [5]–[7]. These systems are aperiodic for an initial amount of time, and then become periodic afterwards. They contain both finite horizon and periodic systems as special cases. What follows is a precise definition of an eventually periodic operator.

Definition 5: An operator P on ℓ_2 is *(h, q)-eventually periodic* if, for some integers $h \geq 0$, $q \geq 1$, we have

$$Z^q((Z^*)^h P Z^h) = ((Z^*)^h P Z^h) Z^q$$

that is P is q-periodic after an initial transient behavior up to time h.

Theorem 6: Suppose that A, B, and C are (h, q)-eventually periodic. Then there exist $X, Y \in \mathcal{X}$ solving inequalities (2) and (3), respectively, if and only if there exist (h, q)-eventually periodic solutions $X_{eper}, Y_{eper} \in \mathcal{X}$.

This is a special case of [5, Th. 12]. Thus, if the system is stable (h, q)-eventually periodic, then we can construct an (h, q)-eventually periodic balanced realization with an (h, q)-eventually periodic diagonal gramian $\Sigma \in \mathcal{X}$ satisfying both Lyapunov inequalities.

Theorem 7: Suppose that system G is an (h, q)-eventually periodic system with a balanced realization (A, B, C, D). Then, the following hold.

i) There exists an (h, q)-eventually periodic diagonal operator $\Sigma \in \mathcal{X}$, partitioned as in (4), satisfying both of the generalized Lyapunov inequalities (2) and (3).

ii) The balanced truncation G_r of G is balanced and satisfies the finite error bound

$$\|G - G_r\| < E_{fh} + 2\sum_i \sigma_i < \infty$$

where σ_i are the *distinct* diagonal entries of the matrix $\mathrm{diag}(\Omega_h, \ldots, \Omega_{h+q-1})$, and E_{fh} is the finite upper bound on the error induced in the balanced truncation of the finite horizon part of G and is derived by applying [2, Th. 2] together with Theorem 3 to attain the tightest bound possible.

Remark 8: The bound on the error induced in the balanced truncation of the periodic part of G is derived in [2], [4], [8], and [9]. We can further employ the model reduction method of [10] to systematically reduce the periodic part of G and potentially obtain a better error bound.

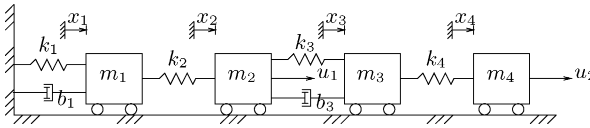

Fig. 1. Translational system.

VI. Example

Consider the four-mass system shown in Fig. 1. The state variable x_i denotes the position of mass m_i with respect to a fixed reference for $i = 1, 2, 3, 4$. The masses are *time-varying* and are assumed to move horizontally on frictionless bearings. The springs and dashpots are linear. The springs are undeflected when $x_i = 0$ for all i. The control inputs are the horizontal forces u_1 and u_2 applied to masses m_2 and m_4, respectively. Suppose the measurable outputs are only positions x_1 and x_3, then the continuous-time state–space realization of this system is given by

$$\dot{x}(t) = A_c(t)x(t) + B_c(t)u(t)$$
$$y(t) = C_c x(t)$$

where the state $x = (x_1, x_2, x_3, x_4, \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)$ and input $u = (u_1, u_2)$ are column vectors, and

$$A_c = \begin{bmatrix} 0_{4\times 4} & I_4 \\ A_{c_{21}} & A_{c_{22}} \end{bmatrix} \quad B_c = \begin{bmatrix} 0_{4\times 2} \\ B_{c_2} \end{bmatrix} \quad C_c = \begin{bmatrix} C_{c_1} & 0_{2\times 4} \end{bmatrix}$$

$$A_{c_{21}} = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{k_3}{m_2} & 0 \\ 0 & \frac{k_3}{m_3} & -\frac{k_3+k_4}{m_3} & \frac{k_4}{m_3} \\ 0 & 0 & \frac{k_4}{m_4} & -\frac{k_4}{m_4} \end{bmatrix}$$

$$A_{c_{22}} = \begin{bmatrix} -\frac{b_1}{m_1} & 0 & 0 & 0 \\ 0 & -\frac{b_3}{m_2} & \frac{b_3}{m_2} & 0 \\ 0 & \frac{b_3}{m_3} & -\frac{b_3}{m_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B_{c_2} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_4} \end{bmatrix}$$

$$C_{c_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The corresponding discrete-time model obtained by zero-order hold sampling is given by

$$x_{k+1} = A_k x_k + B_k u_k$$
$$y_k = C_k x_k$$

where $x_k = x(kT)$, T being the sampling period and k a nonnegative integer, $A_k = \phi((k+1)T, kT)$, $\phi(\cdot, \cdot)$ being the state transition matrix, $C_k = C_c$ for all k, and

$$B_k = \int_{kT}^{(k+1)T} \phi((k+1)T, \tau) B_c(\tau) d\tau.$$

Set the initial state $x_0 = 0$, the sampling period $T = 1$, the spring constants $k_1 = k_3 = 100$ and $k_2 = k_4 = 10^4$, and the friction coefficients $b_1 = b_3 = 10$. We assume that the masses exhibit eventually time-invariant variation, namely each starting at some initial value and then decreasing linearly with time to some final permanent value after, say, $t = 10$. Note that the purpose of this example is to show the usefulness of our error bounds, and for that purpose, we appropriately choose the initial and final values of the masses as follows: $(m_1, m_2, m_3, m_4)_{t=0} = (3, 67, 96, 64)$ and $(m_1, m_2, m_3, m_4)_{t=10} = (1, 39, 24, 3)$. Thus, the resulting discrete-time LTV system is a (10,1)-eventually periodic system of ℓ_2-induced norm equal to 0.4339.

This system is stable and accordingly the corresponding Lyapunov strict inequalities admit positive definite solutions. Since the error bound is clearly dependent on these solutions, we need to make sure that such solutions satisfy some criterion that yields reasonable error bounds, for example, selecting the solution with the minimum trace. Hence, we solve the following semidefinite programming optimization problem:

$$\begin{aligned} \text{minimize} : & \sum_{k=0}^{k=10} (\text{trace } X_k + \text{trace } Y_k) \\ \text{subject to} : & A_k X_k A_k^* - X_{k+1} + B_k B_k^* \prec 0 \\ & A_k^* Y_{k+1} A_k - Y_k + C_k^* C_k \prec 0 \\ & X_k, Y_k \succ 0, \quad \text{for all } k = 0, 1, \ldots, 10 \\ & \text{with } X_{11} = X_{10} \quad Y_{11} = Y_{10}. \end{aligned} \tag{12}$$

Let $X_k = P_k^* P_k$ and $Y_k = Q_k^* Q_k$ be the Cholesky factorizations of the generalized gramians for all $k = 0, 1, \ldots, 10$. Then, performing a singular value decomposition on $Q_k P_k^*$, namely $Q_k P_k^* = U_k \Sigma_k V_k^*$, we compute the balancing state transformation matrix $\mathcal{T}_k$ and its inverse as

$$\mathcal{T}_k = P_k^* V_k \Sigma_k^{-1/2} \qquad \mathcal{T}_k^{-1} = \Sigma_k^{-1/2} U_k^* Q_k$$

and consequently obtain the balanced system realization $(\mathcal{T}_{k+1}^{-1} A_k \mathcal{T}_k, \mathcal{T}_{k+1}^{-1} B_k, C_k \mathcal{T}_k)$ for $k = 0, 1, \ldots, 10$, where $\mathcal{T}_{11} = \mathcal{T}_{10}$.

Since the spring constants $k_2, k_4 \gg k_1, k_3$, it is very likely that four of the states can be truncated at all time instants without inducing any significant error. Indeed, the actual error for such a truncation is $\|G - G_r\| = 4.8140 \times 10^{-4}$, which is about 0.11% of $\|G\|$. Recalling the partition in (4) with Σ given before, the singular values corresponding to the four states that will be truncated are given below in a concise form: $\bar{\Omega} = [\,\omega_0 \quad \omega_1 \quad \cdots \quad \omega_{10}\,] = 10^{-5} \times [\,S_1 \quad S_2\,]$, where

$$S_1 = \begin{bmatrix} 31.727 & 24.405 & 20.625 & 40.284 & 28.086 & 67.603 \\ 24.908 & 15.435 & 17.993 & 26.62 & 27.445 & 30.539 \\ 21.354 & 14.616 & 16.692 & 20.868 & 16.843 & 29.334 \\ 10.91 & 14.39 & 5.2764 & 6.5922 & 13.854 & 6.6306 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 33.993 & 60.91 & 37.019 & 70.889 & 49.923 \\ 31.666 & 36.043 & 36.446 & 44.082 & 47.929 \\ 26.241 & 33.613 & 24.311 & 38.968 & 32.548 \\ 17.548 & 6.6454 & 18.043 & 8.0987 & 9.7647 \end{bmatrix}$$

and each column ω_k is the main diagonal of the corresponding matrix Ω_k. Since we assume zero initial conditions, we can truncate the states at $k = 0$ without inducing any error and, hence, we need not account for the singular values of the first column w_0 when computing the error bound. The last column w_{10} corresponds to the time-invariant part of the system, and the error bound associated with the truncation of states in this part is given by the standard "twice the sum of the tail" formula, namely $2 \times (49.923 + 47.929 + 32.548 + 9.7647) \times 10^{-5} = 2.8033 \times 10^{-3}$. As for the finite horizon part, we only need to consider the singular values of $\hat{\Omega} = (\omega_1, \omega_2, \ldots, \omega_9)$ when computing the error bound, which we will do using both the method of [2] and that of this note.

First, there are no equal singular values in $\hat{\Omega}$, and so we will apply [2, Th. 2] to truncate one state at a time over the finite horizon starting with the last state of the balanced system. For completeness, we will briefly demonstrate how to apply this theorem in such a scenario. So, given a vector of singular values $v = (v_1, v_2, \ldots, v_s)$ for some integer $s \geq 1$, suppose that v_1 cannot be considered as a local maximum and v_s cannot be considered as a local minimum. Then, vector v has m local maximums $v_{\max,i}$ and m local minimums $v_{\min,i}$ for some integer $m \geq 0$, and the upper bound on the error induced in truncating the state corresponding to this vector is $2 \times \mathcal{S}_v$, where $\mathcal{S}_v = v_1$ when $m = 0$, and if $m > 0$, then $\mathcal{S}_v = v_1 \prod_{i=1}^{m} (v_{\max,i}/v_{\min,i})$. Also, sometimes partitioning this vector into smaller ones and applying the preceding argument recursively to each of these vectors might result in tighter error bounds. We will show here how to effectively use the result of [2] to obtain the smallest upper bound possible on the error induced in truncating the eighth state of the balanced system. Note that, in this case, the singular values corresponding to this state are given in the last row of $\hat{\Omega}$. If we are to apply [2, Th. 2] to truncate this state in one step, then the corresponding error bound given by [2] would be

$$2 \times 14.39 \times \frac{13.854}{5.2764} \times \frac{17.548}{6.6306} \times \frac{18.043}{6.6454} \times 10^{-5} \approx 5.43 \times 10^{-3}.$$

This could be improved significantly if we truncate this state in four steps and accordingly divide the bottom row of the matrix $10^5 \times \hat{\Omega}$ into the following four vectors: [14.39], $[\,5.2764 \quad 6.5922 \quad 13.854\,]$, $[\,6.6306 \quad 17.548 \quad 6.6454\,]$, and $[\,18.043 \quad 8.0987\,]$. Then, applying the aforesaid theorem recursively to truncate the state over time intervals corresponding to these vectors, we obtain the following improved bound:

$$2 \times \left(14.39 + 13.854 + 6.6306 \times \frac{17.548}{6.6306} + 18.043\right) \times 10^{-5}$$

which is equal to 1.2767×10^{-3}. This is an improvement by a factor of 4.25. The error bounds given by [2] for truncating the seventh, sixth, and fifth states of the balanced system over the finite horizon are 1.4924×10^{-3}, 8.8164×10^{-4}, and 4.6274×10^{-3}, respectively. Thus, the overall error bound for truncating the last four states over the finite horizon is 8.2782×10^{-3}.

To apply our main result for this finite horizon truncation, we only need to consider the first row in $\hat{\Omega}$. We divide the elements of this row into the following three disjoint sets:

$$\begin{aligned} f_1 &= \{20.625 \times 10^{-5}, 24.405 \times 10^{-5}, 28.086 \times 10^{-5}\} \\ f_2 &= \{33.993 \times 10^{-5}, 37.019 \times 10^{-5}, 40.284 \times 10^{-5}\} \\ f_3 &= \{60.910 \times 10^{-5}, 67.603 \times 10^{-5}, 70.889 \times 10^{-5}\}. \end{aligned}$$

Then, the error bound as given in (6) has the value

$$10^{-5} \cdot \left\{ \left(\sqrt{2}\right)^3 (28.086) + \left(\sqrt{2}\right)^3 (40.284) + \left(\sqrt{2}\right)^3 (70.889) \right\}$$

which is equal to 3.9389×10^{-3}, less than half the value of the error bound given by [2]. If we are to ignore the last three states of the balanced system, and focus on the truncation of the fifth state over the finite horizon, then our error bound ($\approx 3.9 \cdot 10^{-3}$) and that of [2] ($\approx 4.6 \cdot 10^{-3}$) are relatively close. Clearly, the advantage that our method has in this case over that of [2] is that in computing our error bound we do not need to account for the singular values corresponding to the last three states, which happen to contribute significantly to the error bound obtained by applying [2, Th. 2]. This example and others can be found at http://legend.me.uiuc.edu/~mazen/ltv_errbnd/, along with the matlab code used to generate them.

Of course, the preceding example is carefully chosen to highlight the usefulness of our result. In fact, it is quite obvious that if the sequence of Ω_k is large or infinite, then our result might give very conservative bounds, and hence, in general, this result is not to be used exclusively but rather in conjunction with those of [1] and [2]. The most appealing feature about the main result here is that when truncating lots of states over relatively small time intervals, we only need to account for the singular values corresponding to one state when computing the error bound, as demonstrated in the preceding example. This is particularly an advantage over the result of [2] when the singular values excluded in our method contribute significantly to their error bound.

Last, we note that, in computing the error bounds here, solving the optimization problem (12) is the only step that might require heavy computation. As for calculating useful bounds from the singular values, while very simple in this case because of the small finite horizon (see Matlab code), it can become quite challenging in the case of large finite horizons. This is obvious as the results of [1], [2] as well as ours generally give different error bounds depending on how the theorems are applied. This evokes a very interesting research problem, namely developing a fast computational algorithm that effectively applies these results to calculate useful bounds.

VII. Conclusion

This note provides a complementary result to those of [1] and [2]. Its advantages are demonstrated via an example, in which our result gives tighter bounds on the error resulting from the balanced truncation of a four-mass translational system exhibiting eventually periodic dynamics.

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A Solvable Lie Algebra Condition for Stability of Linear Multidimensional Systems

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***Abstract*—This note analyzes exponential stability of a class of linear discrete multidimensional systems. Using a multidimensional comparison principle for estimating the system componentwise exponential convergence and a solvable Lie algebra condition, a sufficient condition for exponential stability of linear multidimensional systems is presented. The stability condition can be easily examined by computing the system matrices in finite steps. This is demonstrated by an example.**

***Index Terms*—Comparison method, exponential stability, multidimensional systems, solvable Lie algebra.**

I. Introduction

Stability and convergence properties of multidimensional systems have been a fundamental problem in theory and applications of control systems and signal processing and attracted considerable attention in the last two decades, e.g., [1]–[11], [13]–[15], [18]–[20] and the references therein. In this note, we study exponential stability of the following linear m-dimensional (m-D) system:

$$x(k) = \sum_{l=1}^{m} A_l x(k - e_l) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the local state indexed by an m-tuple of nonnegative integers $k = (k_1, \ldots, k_m)$, $A_l \in \mathbb{R}^{n\times n}$ is the state matrix, e_l is the lth unit m-vector such that the index $k - e_l = (k_1, \ldots, k_l - 1, \ldots, k_m)$. The initial condition of the system is $x(0) \in \mathbb{R}^n$ for $k = (0, \ldots, 0)$. It is noted that, for the special case $m = 2$, the m-D model (1) is in the homogenous form of the well known Fornasini–Marchesini second model.

The m-D system (1) is called *stable* if for any initial condition $x(0)$, the state $x(k)$ converges asymptotically to zero as $k_1 + \cdots + k_m$ approaches infinity. It is called *exponentially stable* if the state converges at an exponential rate.

It is well known that the stability of a 1-D linear system can be guaranteed by (Schur) stability of its coefficient matrix, i.e., A_1 of the system (1) for the case $m = 1$. However, it has been known that even for a 2-D system in the form (1) with $m = 2$, its stability cannot be easily analyzed, in general, from the system matrices A_1 and A_2.

For a class of m-D systems, it is most desirable to obtain conditions for system stability directly from the system matrices. In [19], it is shown that if the entries of A_1 and A_2 are all nonnegative, then the stability analysis of the 2-D system is greatly reduced to checking the stability of the matrix $A_1 + A_2$. This result was further extended to m-D systems recently in [13] by means of diagonal Lyapunov function argument. The results on nonnegative systems may be of practical interest in certain biological, physical, and economical problems [19].

In this note, we present a result on exponential stability of m-D systems using a comparison method and a solvable Lie algebra condition

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