Proceedings of the 45th IEEE Conference on Decision & Control
Manchester Grand Hyatt Hotel
San Diego, CA, USA, December 13-15, 2006

Control of distributed systems over graphs

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***Abstract*— This paper focuses on designing distributed controllers for interconnected systems in situations where the controller sensing and actuation topology is inherited from that of the plant. The main contribution of this work is new results on general graph interconnection structures. This is accomplished by extending our previous machinery developed for systems with spatial dynamics on $\mathbb{Z}^n$. We derive analysis and convex synthesis conditions for design in this setting. Furthermore, the methodology developed here provides a unifying viewpoint for our previous and related work on distributed control.**

I. Introduction

Distributed control is of considerable interest to the control community because of its importance in large-scale systems, and pursuit of the associated design methods has a long history. Because of the recent emergence of many new applications (e.g., formation flight, antenna arrays, or fluid control) that would benefit from systematic methods for designing distributed controllers, this class of problems has become even more important.

Our goal here is to generalize the work in [2], which was developed for distributed models posed on the standard spatial grid $\mathbb{Z}^n$, to the more general object of graphs. The motivation for this generalization stems from two applications. The first is communication over networks where the plant subsystems have an a priori prescribed communication topology that is not a grid. The second is in connection with control models that arise from finite element approximations, where in general meshes can be arbitrary graphs.

We build on our previous work in [1], [2]. The development here is closely connected with [9] and [7]. In the former, distributed control is considered over groups that are not necessarily abelian, and the latter paper considers distributed control over finite graphs. The framework here is able to capture arbitrary infinite graphs and provides a unifying framework to view previous work. In general, the results obtained here are given in terms of infinite operator inequalities, but can be reduced to semidefinite programs in the cases of infinite graphs with periodic space and time structure or finite graphs with temporal periodicity. One further case in which the results can be stated as semidefinite programming problems is when the spatial and temporal variation is *eventually periodic* as depicted in Figure 1. Work on the purely time-varying case, where the term eventually periodic was introduced, appears in [4]. In this paper, for space considerations, we will only consider the general case, leaving the aforesaid special cases for the journal version.

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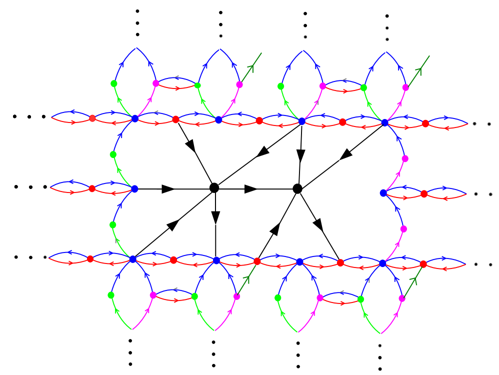


Fig. 1. An eventually periodic graph.

II. Preliminaries

The natural numbers (including zero), integers, integers modulo m and real numbers are denoted by $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Z}_m$ and $\mathbb{R}$ respectively. We use $\mathcal{G}(V, E)$ to denote a digraph with set of vertices V and set of directed edges E; throughout we will assume that the number of vertices is countable and that the vertex degree is uniformly bounded. We use the ordered-pair (i, j) to denote the element of E corresponding to an edge directed from vertex $i \in V$ to vertex $j \in V$. We will use $s(\mathcal{G})$ to denote the maximum over the indegrees and outdegrees of the graph; namely,

$$s(\mathcal{G}) := \max_{k \in V}\{m(k), p(k)\},$$

where $m(k)$ and $p(k)$ denote the indegree and outdegree of a vertex $k \in V$ respectively. We will call a bijection $\sigma : V \to V$ a *permutation* of V. A subset $\mathcal{C}$ of V is a *cycle* of the permutation σ if there exist an integer $1 \leq m \leq \infty$ and a bijection $\beta : \mathcal{C} \to \mathbb{Z}_m$ such that

$$\beta \circ \sigma(v) = (1 + \beta(v)) \bmod m.$$

Note that here by $m = \infty$ we mean $\mathbb{Z}_m$ is simply the integers $\mathbb{Z}$. It is easy to show that the family of cycles of σ, denoted $\mathcal{C}_i$, are countable in number, disjoint, and that $V = \bigcup_i \mathcal{C}_i$.

We also define the set $\mathbb{G}$ as

$$\mathbb{G} = \mathbb{Z} \times V = \{\bar{k} = (t, k) \mid t \in \mathbb{Z} \text{ and } k \in V\}.$$

If we have two vectors of real numbers x and y, the notation $x \leq y$ will be used to mean that the inequalities $x_j \leq y_j$ hold for all indices j; if x and y are infinite sequences, we adopt a similar pointwise meaning. The maximum singular value of a matrix M will be denoted by $\bar{\sigma}(M)$. Given a symmetric matrix H, its inertia $\text{in}(H)$ is the triplet $(\text{in}_+(H), \text{in}_0(H), \text{in}_-(H))$ giving the number of positive,

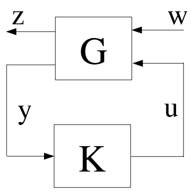


Fig. 2. Closed-loop system

zero, and negative eigenvalues of H, respectively. If H is of zero dimensions, then we set $\mathrm{in}(H) = (0, 0, 0)$. We now state the following result.

Proposition 1: Suppose H is an $n \times n$ symmetric matrix, and that Q is an $n \times m$ matrix. Then $\mathrm{in}_+(H) \geq \mathrm{in}_+(Q^* HQ)$ and $\mathrm{in}_-(H) \geq \mathrm{in}_-(Q^* HQ)$. Furthermore, if $m = n$ and Q is nonsingular, then $\mathrm{in}(H) = \mathrm{in}(Q^* HQ)$.

If V is a vector space, we will say that the linear mapping $M : V \to V$ has an *algebraic* inverse on V if there exists another linear mapping on V, denoted M^{-1}, such that both MM^{-1} and $M^{-1}M$ are equal to the identity map. Given a Hilbert space H, we denote its associated norm by $\|\cdot\|_H$ and its inner product by $\langle \cdot, \cdot \rangle_H$; for convenience we frequently suppress the subscript. The notation $H \oplus W$ will refer to the Hilbert space direct sum of the spaces H and W. Given two Hilbert spaces H and F, we denote the space of bounded linear operators mapping H to F by $\mathcal{L}(H, F)$, and shorten this to $\mathcal{L}(H)$ when H equals F. If X is in $\mathcal{L}(H, F)$, we denote the H to F induced norm of X by $\|X\|_{H \to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. An operator $X \in \mathcal{L}(H, F)$ is *coercive* if there exists an $\alpha > 0$ such that $\|Xu\|_F \geq \alpha \|u\|_H$ holds for all u in H. When an operator $X \in \mathcal{L}(H)$ is self-adjoint, we use $X \prec 0$ to mean it is *negative definite*; that is there exists a number $\alpha > 0$ such that, for all nonzero $x \in H$, the inequality $\langle x, Xx \rangle < -\alpha \|x\|^2$ holds.

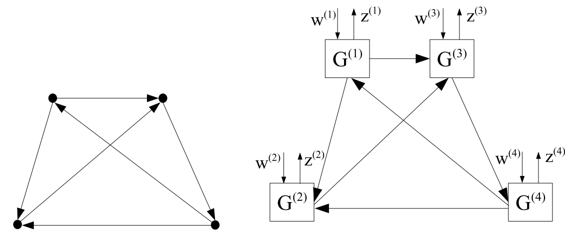
Suppose that $n(\bar{k})$ is an integer sequence mapping $\mathbb{G}$ to the nonnegative integers $\mathbb{N}_0$. We define $\ell(\{\mathbb{R}^{n(\bar{k})}\})$ to be the vector space of mappings w which satisfy $w : \bar{k} \in \mathbb{G} \mapsto w(\bar{k}) \in \mathbb{R}^{n(\bar{k})}$. We will frequently abbreviate this notation to simply ℓ when the dimensions are clear from the context. We will use $\ell_2(\{\mathbb{R}^{n(\bar{k})}\})$ to denote the subspace of $\ell(\{\mathbb{R}^{n(\bar{k})}\})$ which is a Hilbert space under the norm

$$\|w\|_2 := \left(\sum_{\bar{k} \in \mathbb{G}} |w(\bar{k})|_2^2 \right)^{\frac{1}{2}},$$

where $|\cdot|_2$ is the Euclidean norm. Furthermore, we define $\ell_{2e}(\{\mathbb{R}^{n(\bar{k})}\})$ to be the subset of ℓ satisfying for each fixed $t \in \mathbb{Z}$ the inequality $\sum_{k \in V} |w(t,k)|_2^2 < \infty$. In other words, $w(t, \cdot)$ is in an ℓ_2 space for each t. Of the three spaces ℓ, ℓ_2 and ℓ_{2e} just defined, we will be for most part dealing with $\ell_2(\{\mathbb{R}^{n(\bar{k})}\})$ in the sequel.

III. Distributed system formulation

This paper deals with distributed systems, whose interconnection structures are defined by directed graphs. Specifically, the different linear time-varying systems comprising the distributed model in question correspond to the vertices of the graph, and the interconnections between these systems are described by the directed edges of the graph. Given such a distributed system, say G, we will consider in this paper a controller *synthesis* problem, illustrated in Figure 2, where the controller K has a similar structure to G.



(a) (b)

Fig. 3. Graph and system

Consider a digraph $\mathcal{G}(V, E)$, which defines the information structure of the distributed system under consideration, and so, for instance, graph (a) in Figure 3 gives rise to the interconnected system in (b). Henceforth, we will conveniently use the term vertex to also refer to a constituent system in a distributed model. Our goal now is to present general state-space equations which describe each of the vertices of the distributed model. For convenience of notation, we set $s = s(\mathcal{G})$. Now, let σ_1, σ_2, ..., σ_s be permutations on the vertices V. We choose these with the property that if $(i, j) \in E$, then there exists r such that $\sigma_r(i) = j$, and hence $\sigma_r^{-1}(j) = i$ because of the bijective correspondence.

Example: Consider the distributed system in Figure 4(a). We can use the graph in Figure 4(b) to represent this system. Here, $s = \max_k \{m(k), p(k)\} = \max\{2, 2\} = 2$, and we can define the permutations σ_1 and σ_2 for each vertex k as

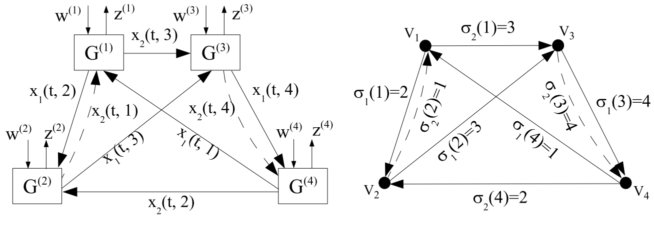
$k=1$	$k=2$	$k=3$	$k=4$
$\sigma_1(1) = 2$	$\sigma_1(2) = 3$	$\sigma_1(3) = 4$	$\sigma_1(4) = 1$
$\sigma_2(1) = 3$	$\sigma_2(2) = 1$	$\sigma_2(3) = 4$	$\sigma_2(4) = 2$

We will regard the interconnections between the systems $G^{(k)}$ as states when formulating the system equations, and accordingly we define our state-space model as follows:

$$\begin{bmatrix} x_0(t+1, k) \\ x_1(t, \sigma_1(k)) \\ \vdots \\ x_s(t, \sigma_s(k)) \end{bmatrix} = A(t,k) \begin{bmatrix} x_0(t,k) \\ x_1(t,k) \\ \vdots \\ x_s(t,k) \end{bmatrix} + B(t,k) \begin{bmatrix} w(t,k) \\ u(t,k) \end{bmatrix}$$

$$\begin{bmatrix} z(t,k) \\ y(t,k) \end{bmatrix} = \begin{bmatrix} C_1(t,k) \\ C_2(t,k) \end{bmatrix} x(t,k) + \begin{bmatrix} D_{11}(t,k) & D_{12}(t,k) \\ D_{21}(t,k) & D_{22}(t,k) \end{bmatrix} \begin{bmatrix} w(t,k) \\ u(t,k) \end{bmatrix} \quad (1)$$

where $t \in \mathbb{Z}$ is discrete-time and k is any vertex in V. The input signals w and u denote the exogenous disturbances and applied control, respectively, whereas the output signal z denotes the exogenous errors and y the measurements. Notice that, in the previous equations, the state vector $x(t, k)$ is partitioned into $s+1$ separate vector-valued channels, namely $x(t, k) = (x_0(t, k), \ldots, x_s(t, k))$. Then, comfortably with these channels, we partition the state-space matrices as



(a) (b)

Fig. 4. A distributed system over a digraph

$$A(\bar{k}) = \begin{bmatrix} A_{00}(\bar{k}) & \cdots & A_{0s}(\bar{k}) \\ \vdots & \ddots & \vdots \\ A_{s0}(\bar{k}) & \cdots & A_{ss}(\bar{k}) \end{bmatrix}, \; B(\bar{k}) = \begin{bmatrix} B_{01}(\bar{k}) & B_{02}(\bar{k}) \\ \vdots & \vdots \\ B_{s1}(\bar{k}) & B_{s2}(\bar{k}) \end{bmatrix}$$
$$\text{and } C(\bar{k}) = \begin{bmatrix} C_{10}(\bar{k}) & \cdots & C_{1s}(\bar{k}) \\ C_{20}(\bar{k}) & \cdots & C_{2s}(\bar{k}) \end{bmatrix}.$$

Throughout this work, we assume that these matrix sequences are uniformly bounded. We will allow matrix dimensions that depend on (t, k), and thus define the sequences $c(t, k)$, $n_r(t, k)$, $n_c(t, k)$, and $b(t, k)$ so that, for each $(t, k) \in \mathbb{Z} \times V$, we have that $C(t, k)$ is $c(t, k) \times n_c(t, k)$, $A(t, k)$ is $n_r(t, k) \times n_c(t, k)$, and $B(t, k)$ is $n_r(t, k) \times b(t, k)$. For now, we will assume that this model admits a unique solution x in ℓ_{2e} given inputs $w, u \in \ell_{2e}$, and that the associated mapping $(w, u) \mapsto x$ is causal; in the next section, we will give precise conditions to ensure these properties.

It is important to point out that Figures 3 and 4 represent the same system; notice however that there are two extra edges represented by dashed arrows in Figure 4. These virtual edges are added because, for our notation to work, the indegree and outdegree of each vertex has to be equal to s. This is necessary so that it is possible to define the permutations $\sigma_1, \ldots, \sigma_s$; recall that each σ_i has to be defined over the entire vertex set. Now, since the states $x_2(t, 4)$ and $x_2(t, 1)$ correspond to non-existent interconnections, we set the row dimensions of these states equal zero; also the associated blocks in the state-space matrices will appropriately have zero row or column dimensions. Although a slight abuse of notation, this will allow for a succinct representation, and all the manipulations and proofs in the sequel will follow directly, namely, when we encounter matrix blocks of zero dimensions, say in some operator inequality corresponding to some vertex, this just means that the rows and columns of which these blocks are elements are not present, and the inequalities remain valid. It is worth noting, furthermore, that in some instances, especially those involving semi-infinite vertex arrays, it might be necessary to add virtual vertices along with virtual edges for the notation to work.

Notice that if we define $\sigma_r(i) = j$, corresponding to edge (i, j), then the associated state will be denoted by $x_r(t, j)$. Last, as an example, we write the system equation of $G^{(3)}$:

$$\begin{bmatrix} x_0(t+1, 3) \\ x_1(t, \sigma_1(3) = 4) \\ x_2(t, \sigma_2(3) = 4) \\ z(t, 3) \end{bmatrix} = \begin{bmatrix} A(t, 3) & B(t, 3) \\ C(t, 3) & D(t, 3) \end{bmatrix} \begin{bmatrix} x_0(t, 3) \\ x_1(t, 3) \\ x_2(t, 3) \\ w(t, 3) \end{bmatrix},$$

where $x_2(t, 4)$ has zero row dimension. □

A. Graph-diagonal operators

The next step is to develop operator theoretic machinery a la [2], which will allow us to represent the system equations of (1) in a compact operator form.

Definition 2: Let v and n be sequences mapping $\mathbb{G}$ to $\mathbb{N}_0$, and Q be a linear mapping from $\ell_2(\{\mathbb{R}^{v(\bar{k})}\})$ to $\ell_2(\{\mathbb{R}^{n(\bar{k})}\})$. Then Q is said to be a graph-diagonal operator if there exists a uniformly bounded sequence of matrices $Q(\bar{k}) \in \mathbb{R}^{n(\bar{k}) \times v(\bar{k})}$ such that the equality $(Qw)(\bar{k}) = Q(\bar{k})w(\bar{k})$ holds for each $\bar{k} \in \mathbb{G}$.

Graph-diagonal operators are defined similarly to hyperdiagonal operators, considered in [2], and both are generalizations of block-diagonal operators.

As in the case of hyperdiagonal operators in [2] and block-diagonal operators in [3], we generalize the concept of inertia to graph-diagonal operators. Given a self-adjoint graph-diagonal operator Q, we define its inertia to be the mapping

$$\text{In}(Q) : \mathbb{G} \to \mathbb{N}_0^3 \;\text{ defined by }\; \text{In}(Q)(\bar{k}) := \text{in}(Q(\bar{k})).$$

Similarly, we define $\text{In}_+(Q)(\bar{k}) := \text{in}_+(Q(\bar{k}))$ and $\text{In}_-(Q)(\bar{k}) := \text{in}_-(Q(\bar{k}))$. Then, Proposition 1 generalizes immediately to the following congruence result for graph-diagonal operators.

Proposition 3: Suppose H and M are graph-diagonal operators, with H self-adjoint. Then $\text{In}_+(H) \geq \text{In}_+(M^*HM)$ and $\text{In}_-(H) \geq \text{In}_-(M^*HM)$. Furthermore, if M is nonsingular, then $\text{In}(H) = \text{In}(M^*HM)$.

We say that W is a *partitioned* graph-diagonal operator if it has the form

$$W = \begin{bmatrix} H & P \\ R & J \end{bmatrix},$$

where H, P, R and J are graph-diagonal operators. We then define the *graph-diagonal representation* of W as the graph-diagonal operator $[\![W]\!]$ given by:

$$\left([\![W]\!]x\right)(\bar{k}) := \begin{bmatrix} H(\bar{k}) & P(\bar{k}) \\ R(\bar{k}) & J(\bar{k}) \end{bmatrix} x(\bar{k}).$$

Clearly these concepts generalize to arbitrary partitions. Given vector-valued sequences $\bar{q} = (q_1, \ldots, q_r) : \mathbb{G} \to \mathbb{N}^r$ and $\bar{v} = (v_1, \ldots, v_c) : \mathbb{G} \to \mathbb{N}^c$, we denote by $\mathcal{P}(\bar{q}, \bar{v})$ the set of partitioned graph-diagonal operators of the form

$$\begin{bmatrix} J_{11} & \cdots & J_{1c} \\ \vdots & \ddots & \vdots \\ J_{r1} & \cdots & J_{rc} \end{bmatrix}, \qquad (2)$$

where each J_{ij} is a graph-diagonal operator mapping $\ell_2(\{\mathbb{R}^{v_j(\bar{k})}\})$ to $\ell_2(\{\mathbb{R}^{q_i(\bar{k})}\})$. The following notation will be convenient: given a partitioned graph-diagonal operator J in $\mathcal{P}(\bar{q}, \bar{v})$, we define $p(J) := (\bar{q}, \bar{v})$. Furthermore, if $\bar{q}$ and $\bar{v}$ are dependent on each other (e.g., $\bar{q} = \bar{v}$), we will simply set $p(J) = \bar{q}$ and write $\mathcal{P}(\bar{q})$ in place of $\mathcal{P}(\bar{q}, \bar{v})$. Also, when the partition dimensions $(\bar{q}, \bar{v})$ are not important, we will use the abbreviation $\mathcal{P}$ to denote the set of partitioned graph-diagonal operators.

It is not difficult to see that $[\![\cdot]\!]$ is a homomorphism from $\mathcal{P}$ into the space of graph-diagonal operators, which is isometric, and preserves products, addition, and ordering. We define the inertia of a self-adjoint, partitioned graph-diagonal operator W by $\text{In}(W) := \text{In}([\![W]\!])$. Clearly the definition of graph-diagonal operators is extendable to ℓ and ℓ_{2e}, and in the sequel, we will not distinguish between these objects. We now define a number of shift operators. For some sequence $n(\bar{k})$, we define the time shift or delay operator S_0 by

$$S_0 : \ell(\{\mathbb{R}^{n(\bar{k})}\}) \to \ell(\{\mathbb{R}^{q(\bar{k})}\}),\ \text{where } q(t+1,k) = n(t,k),$$
$$(S_0 v)(t,k) = v(t-1,k),$$

and further define, for $i = 1, 2, \ldots, s$, the spatial shift operators S_i by

$$S_i : \ell(\{\mathbb{R}^{n(\bar{k})}\}) \to \ell(\{\mathbb{R}^{q(\bar{k})}\}),\ \text{where } q(t,\sigma_i(k)) = n(t,k),$$
$$(S_i v)(t,k) = v(t,\sigma_i^{-1}(k)).$$

Clearly, these shifts are invertible, and it is easy to verify that $(S_0^{-1}v)(t,k) = v(t+1,k)$ and $(S_i^{-1}v)(t,k) = v(t,\sigma_i(k))$ for $i = 1, 2, \ldots, s$. In fact, each of these shifts is also unitary, and so the inverse is equal to the adjoint. We also define the *composite* shift operator $S := \text{diag}(S_0, S_1, \ldots, S_s)$.

Before stating the desired equations, notice that the matrix sequences $A_{ij}(\bar{k})$, $B_{ir}(\bar{k})$, $C_{lj}(\bar{k})$ and $D_{lr}(\bar{k})$ in (1) define partitioned graph-diagonal operators A, B, C and D. Then, we can equivalently write the state-space equations (1) in compact operator form as

$$x = SAx + SB\begin{bmatrix} w \\ u \end{bmatrix}, \qquad \begin{bmatrix} z \\ y \end{bmatrix} = Cx + D\begin{bmatrix} w \\ u \end{bmatrix}. \quad (3)$$

Then, assuming the relevant algebraic inverse exists, we can rewrite (3) in the linear fractional form

$$G = C(I - SA)^{-1}SB + D; \quad (4)$$

here G designates the input-output mapping of the model (1).

Before concluding this section, it is convenient to partition the composite shift and state-space operators in (3) along temporal-spatial lines. Namely, let us partition the shift as $S = \text{diag}(S_0, \bar{S})$, where $\bar{S} = \text{diag}(S_1, \ldots, S_s)$, and compatibly partition the state-space operators so that

$$A = \begin{bmatrix} A_{00} & \bar{A}_{0\bullet} \\ \bar{A}_{\bullet 0} & \bar{A} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_0 \\ \bar{B} \end{bmatrix}. \quad (5)$$

IV. Analysis results

This section defines the basic concepts of well-posedness and stability for distributed systems in the form of (1). Also, a version of the Kalman-Yakubovich-Popov (KYP) Lemma is given for such systems, which gives sufficient conditions for stability and contractiveness of open-loop systems.

Definition 4: A distributed model in the form of (1) is well-posed if, given inputs in ℓ_{2e}, the equations (1) admit unique solutions $x_i \in \ell_{2e}$ for $i = 0, 1, \ldots, s$, and furthermore define a linear causal mapping on ℓ_{2e}.

Regarding the unique solutions in ℓ_{2e}, this can always be guaranteed if the operator $I - SA$ has an algebraic inverse on $\oplus_{j=0}^{s}\ell_{2e}$, as evident from (4). Moreover, causality is ensured if this inverse is causal; this also follows from (4) since the other operators in this equation are causal and the space of causal operators is an algebra.

Lemma 5: The operator $I - SA$ has an algebraic causal inverse on $\oplus_{j=0}^{s}\ell_{2e}$, and hence the system in (1) is well-posed, if (i) $A(t,k)$, and all the other state-space matrices, are equal to zero when $t < 0$; and (ii) the linear mapping $I - \bar{S}\bar{A}$ has an algebraic inverse on $\oplus_{j=1}^{s}\ell_{2e}$, where $\bar{S}$ and $\bar{A}$ correspond to the partition in (5).

The proof is basically the same as that of [2, Lemma 8].

Definition 6: We say the system in (1) is stable if it is well-posed and, given inputs in ℓ_2, the equations (1) admit unique solutions $x_i \in \ell_2$ for $i = 0, 1, \ldots, s$, and further define a linear causal mapping on ℓ_2.

In other words, the system is stable if $I - SA$ has a causal inverse on $\oplus_{j=0}^{s}\ell_2$. This ensures that G from (4) is a linear causal map on ℓ_2. We will formulate next Lyapunov-based tests for stability.

At this point, it is important to state some important properties about shifts and graph-diagonal operators. To begin, suppose that W is a graph-diagonal operator on ℓ_2. Then both $S_i^* W S_i$ and $S_i W S_i^*$ are also graph-diagonal, and given by the relationships

$$(S_0^* W S_0)(t,k) = W(t+1,k),\ (S_0 W S_0^*)(t,k) = W(t-1,k),$$
$$(S_j^* W S_j)(t,k) = W(t,\sigma_j(k)),\ (S_j W S_j^*)(t,k) = W(t,\sigma_j^{-1}(k)),$$

for $j = 1, \ldots, s$. If we define $X \in \mathcal{P}$ to be block-diagonal with respect to its partition (also that of S), i.e. $X = \text{diag}(X_0, X_1, \ldots, X_s)$, then S^*XS and SXS^* are in $\mathcal{P}$ as well. We now define the subset $\mathcal{X}$ of $\mathcal{P}$ as

$$\mathcal{X} = \{X \in \mathcal{P} : X = \text{diag}(X_0, X_1, \ldots, X_s),$$
$$X^{-1} \in \mathcal{L}(\oplus_{j=0}^{s}\ell_2),\ \text{and}\ \ X_0 \succ 0\}.$$

Lemma 7: Given $A \in \mathcal{P}$, if there exists an operator $X \in \mathcal{X}$ satisfying $\text{In}_-(S^*XS) = \text{In}_-(X)$ and $S^*A^*XSA - X \prec 0$, then $I - SA$ has a causal inverse on $\oplus_{j=0}^{s}\ell_2$, and hence the system in (1) is stable.

The proof is formally the same as that of [2, Lemma 13].

The above result provides a sufficient condition for stability, which in general is not necessary and so some conservatism is introduced into our analysis.

Lemma 8: Suppose X is in $\mathcal{X}$ and satisfies the inertia condition $\text{In}_-(S^*XS) = \text{In}_-(X)$. If the inequality

$$\begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \prec 0 \quad (6)$$

holds, then $I - SA$ has a causal inverse on $\oplus_{j=0}^{s}\ell_2$ and $\|C(I - SA)^{-1}SB + D\| < 1$.

The proof is formally identical to that of [2, Lemma 14].

V. Synthesis

This section tackles the controller synthesis problem for the distributed system in (1).

Definition 9: A controller K is an *admissible* synthesis for plant G in Figure 2 if it ensures a stable closed-loop system and achieves the inequality $\|w \mapsto z\| < 1$.

To start, the system realization (A, B, C, D) from (3) represents the distributed system in question. To ensure well-posedness, we assume that this realization satisfies the conditions (i) and (ii) in Lemma 5. We also assume for convenience that $D_{22} = 0$. Suppose this system is being controlled by a controller K with realization (A_K, B_K, C_K, D_K), and state dimensions given by the sequence $\bar{n}_K$, i.e. $\bar{n}_K = p(A_K)$. The closed-loop system in Figure 2 can then have the representation $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$, where

$$A_{cl} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix},\ \ B_{cl} = \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix},$$
$$C_{cl} = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix},\ D_{cl} = D_{11} + D_{12} D_K D_{21}.$$

Clearly, the state dimensions of this closed-loop realization are given by $\bar{n}_{cl} := (p(A), \bar{n}_K)$. Defining $S_{cl} = \mathrm{diag}(S, S)$, we can write the closed-loop equations in the fractional form $w \mapsto z = C_{cl}(I - S_{cl}A_{cl})^{-1}S_{cl}B_{cl} + D_{cl}$.

Let X_{cl} be a partitioned graph-diagonal operator of the form

$$X_{cl} = \begin{bmatrix} X & X_{GK} \\ X_{GK}^* & X_K \end{bmatrix} \in \mathcal{P}(\bar{n}_{cl}), \tag{7}$$

where $X \in \mathcal{X}$, $X_{GK} = \mathrm{diag}(X_{GK0}, \ldots, X_{GKs}) \in \mathcal{P}(p(A); \bar{n}_K)$, and $X_K = \mathrm{diag}(X_{K0}, \ldots, X_{Ks}) \in \mathcal{P}(\bar{n}_K)$. We define the set $\mathcal{X}_{cl}$ as

$$\mathcal{X}_{cl} := \Big\{X_{cl} \in \mathcal{P}(\bar{n}_{cl}) : X_{cl} = X_{cl}^* \text{ partitioned as in (7)}, \; X_{cl}^{-1} \text{ exists, and } \begin{bmatrix} X_0 & X_{GK0} \\ X_{GK0}^* & X_{K0} \end{bmatrix} \succ 0\Big\}.$$

Lemma 10: A controller K is an admissible synthesis for the configuration in Fig 2 if there exists an operator X_{cl} in $\mathcal{X}_{cl}$ such that $\mathrm{In}_-(X_{cl}) = \mathrm{In}_-(S_{cl}^* X_{cl} S_{cl})$ and

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}^* \begin{bmatrix} S_{cl}^* X_{cl} S_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} - \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{8}$$

Notice that, from the definition of $\mathcal{X}_{cl}$, it is obvious that $S_{cl}^* X_{cl} S_{cl}$ is a partitioned graph-diagonal operator.

The game plan now to to transform the conditions of the above lemma into convex ones which are only dependent on the plant data. The approach we employ is analogous to the ones used in [6], [8], and particularly parallels that of [2]. We commence with parameterizing the closed-loop realization:

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = R + U^* Q V, \text{ where } Q = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}. \tag{9}$$

Note that the partitioned graph-diagonal operators R, U, and V depend *only* on the plant. We next exploit this parametrization to rid the conditions of Lemma 10 from the controller data. But first, we need to make the following definition. Given symbols $E_1, \ldots, E_4$, Q_1, Q_2, and N, we define the notation $\mathbf{L}\left(\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}, Q_1, Q_2, N\right) :=$

$$N^* \left\{ \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}^* \begin{bmatrix} Q_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} - \begin{bmatrix} Q_2 & 0 \\ 0 & I \end{bmatrix} \right\} N.$$

Proposition 11: Suppose that $X_{cl} \in \mathcal{X}_{cl}$ and satisfies $\mathrm{In}_-(X_{cl}) = \mathrm{In}_-(S_{cl}^* X_{cl} S_{cl})$, and consider the partitioned graph-diagonal operators $X, Y \in \mathcal{X}$ defined by

$$X_{cl} = \begin{bmatrix} X & X_{GK} \\ X_{GK}^* & X_K \end{bmatrix} \text{ and } X_{cl}^{-1} = \begin{bmatrix} Y & Y_{GK} \\ Y_{GK}^* & Y_K \end{bmatrix}. \tag{10}$$

Then, inequality (8) holds if and only if

$$\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, S^* X S, X, N_X\right) \prec 0 \text{ and}$$

$$\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^*, Y, S^* Y S, N_Y\right) \prec 0 \text{ both hold,} \tag{11}$$

where N_X and N_Y are coercive operators with

$$\mathrm{Im}\, N_X = \mathrm{Ker}\begin{bmatrix} C_2 & D_{21} \end{bmatrix} \text{ and } \mathrm{Im}\, N_Y = \mathrm{Ker}\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}.$$

The proof is very similar to that of [2, Proposition 20].

We still require some additional convex conditions on X and Y such that the validity of these conditions ensures the existence of a corresponding X_{cl} of the desired properties. To this end, we introduce the following.

Lemma 12: Suppose n and h are sequences of nonnegative integers, and that X and Y are positive definite graph-diagonal operators in $\mathcal{P}(n)$. Then, there exist operators X_2, Y_2 in $\mathcal{P}(n, h)$, and self-adjoint operators X_3, $Y_3 \in \mathcal{P}(h)$ satisfying

$$\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \succ 0 \text{ and } \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix} \tag{12}$$

if and only if

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \text{ and } \mathrm{In}_+\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right) \leq n + h. \tag{13}$$

Proof: Conditions (13) hold if and only if, for all $\bar{k} \in \mathbb{G}$, we have

$$\begin{bmatrix} X(\bar{k}) & I \\ I & Y(\bar{k}) \end{bmatrix} \succeq 0, \; \mathrm{in}_+\left(\begin{bmatrix} X(\bar{k}) & I \\ I & Y(\bar{k}) \end{bmatrix}\right) \leq n(\bar{k}) + h(\bar{k}).$$

This corresponds to the case $i_- = 0$ in [2, Lemma 21], and thus, by this lemma, these conditions hold if and only if there exist matrix sequences with $X_2(\bar{k}), Y_2(\bar{k}) \in \mathbb{R}^{n(\bar{k}) \times h(\bar{k})}$ and symmetric matrices $X_3(\bar{k}), Y_3(\bar{k}) \in \mathbb{R}^{h(\bar{k}) \times h(\bar{k})}$ such that

$$\begin{bmatrix} X(\bar{k}) & X_2(\bar{k}) \\ X_2^*(\bar{k}) & X_3(\bar{k}) \end{bmatrix}^{-1} = \begin{bmatrix} Y(\bar{k}) & Y_2(\bar{k}) \\ Y_2^*(\bar{k}) & Y_3(\bar{k}) \end{bmatrix} \succ 0. \tag{14}$$

Furthermore, these sequences can be chosen so that

$$\bar{\sigma}\left(\begin{bmatrix} X(\bar{k}) & X_2(\bar{k}) \\ X_2^*(\bar{k}) & X_3(\bar{k}) \end{bmatrix}\right) \leq \|X\| + \|X - Y^{-1}\|^{\frac{1}{2}} + 1 \text{ and}$$

$$\bar{\sigma}\left(\begin{bmatrix} X(\bar{k}) & X_2(\bar{k}) \\ X_2^*(\bar{k}) & X_3(\bar{k}) \end{bmatrix}^{-1}\right) \leq \|Y\| \cdot (1 + \|X - Y^{-1}\|^{\frac{1}{2}})^2 + 1.$$

This is equivalent to saying that, for all $\bar{k} \in \mathbb{G}$, there exist positive scalars α and β such that

$$\alpha I \succ \left[\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix} \right]\!\right](\bar{k}) \succ \beta I,$$

where X_2, X_3 are graph-diagonal operators constructed from the associated matrices in the obvious way. This inequality, along with (14), is equivalent to (12). ∎

At this point, it is convenient to define the following notation. Given an index $1 \leq j \leq s$, we define $\mathcal{C}^{(j)}$ as the family of cycles of σ_j, and we denote the ith cycle in this family by $\mathcal{C}_i^{(j)}$, i.e. $\mathcal{C}^{(j)} = \{\mathcal{C}_i^{(j)}\}$. With this said, given a partitioned graph-diagonal operator $W \in \mathcal{P}$, an index $1 \leq j \leq s$, $t \in \mathbb{Z}$, and $\mathcal{C}^{(j)} = \{\mathcal{C}_i^{(j)}\}$, we define

$$\overline{\mathrm{In}}_-^j(W)(t, \mathcal{C}_i^{(j)}) := \max_{k \in \mathcal{C}_i^{(j)}} \left(\mathrm{in}_-\{[\![W]\!](t, k)\}\right).$$

We can now state the following important result.

Lemma 13: Suppose j is an integer in $\{1, \ldots s\}$, that $n(\bar{k})$ and $h(\bar{k})$ are nonnegative integer sequences, and that X and Y are in $\mathcal{P}(n)$. Then, there exist X_2, Y_2 in $\mathcal{P}(n, h)$, and self-adjoint operators X_3, $Y_3 \in \mathcal{P}(h)$ satisfying

$$\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix} \text{ and}$$

$$\mathrm{In}_-\left(\left[\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix} \right]\!\right]\right) = \mathrm{In}_-\left(S_j^* \left[\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix} \right]\!\right] S_j\right) \tag{15}$$

if and only if, for each $\mathcal{C}_i^{(j)} \in \mathcal{C}^{(j)}$ and $t \in \mathbb{Z}$,

$$\mathrm{In}_+\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right)(t,k) + \overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right)(t,\mathcal{C}_i^{(j)}) \leq n(t,k)+h(t,k) \text{ holds for all } k \in \mathcal{C}_i^{(j)}. \quad (16)$$

Proof: Observe that condition (15) is equivalent to

$$\mathrm{in}_-\left(\left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right](t,k)\right) = \mathrm{in}_-\left(\left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right](t,\sigma_j(k))\right) \quad (17)$$

for all $(t,k) \in \mathbb{Z} \times V$.

(Only if): From equation (17), it is immediate that, for every $t \in \mathbb{Z}$ and $\mathcal{C}_i^{(j)} \in \mathcal{C}^{(j)}$, the equation

$$\overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right)(t,\mathcal{C}_i^{(j)}) = \mathrm{in}_-\left(\left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right](t,k)\right)$$

holds for all $k \in \mathcal{C}_i^{(j)}$, and so, we have $n(t,k)+h(t,k) =$

$$\mathrm{In}_+\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right)(t,k) + \overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right)(t,\mathcal{C}_i^{(j)}) \quad (18)$$

for all $(t,k) \in \mathbb{Z} \times \mathcal{C}_i^{(j)}$. We can write the factorization

$$\begin{bmatrix} I & 0 \\ Y & Y_2 \end{bmatrix}^* \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & Y_2 \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}.$$

Then, by Proposition 3, we get

$$\overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right)(t,\mathcal{C}_i^{(j)}) \geq \overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right)(t,\mathcal{C}_i^{(j)})$$

$$\mathrm{In}_+\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right) \geq \mathrm{In}_+\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right)$$

for all $(t,k) \in \mathbb{Z} \times \mathcal{C}_i^{(j)}$, and so the result follows from (18).

(If): Fix $t \in \mathbb{Z}$, $\mathcal{C}_i^{(j)} \in \mathcal{C}^{(j)}$. Then, from (16), there exist a nonnegative integer q_- and a sequence $q_+(k)$ such that, for all $k \in \mathcal{C}_i^{(j)}$, we have $n(t,k)+h(t,k) = q_+(k)+q_-$, and

$$\mathrm{in}_-\left(\left[\!\!\left[\begin{matrix} X & I \\ I & Y \end{matrix}\right]\!\!\right](t,k)\right) \leq q_-, \ \mathrm{in}_+\left(\left[\!\!\left[\begin{matrix} X & I \\ I & Y \end{matrix}\right]\!\!\right](t,k)\right) \leq q_+(k).$$

Then, by [2, Lemma 21], there exist matrix sequences with $X_2(\bar{k}), Y_2(\bar{k}) \in \mathbb{R}^{n(\bar{k}) \times h(\bar{k})}$ and symmetric matrices $X_3(\bar{k}), Y_3(\bar{k}) \in \mathbb{R}^{h(\bar{k}) \times h(\bar{k})}$ such that

$$\begin{bmatrix} X(\bar{k}) & X_2(\bar{k}) \\ X_2^*(\bar{k}) & X_3(\bar{k}) \end{bmatrix}^{-1} = \begin{bmatrix} Y(\bar{k}) & Y_2(\bar{k}) \\ Y_2^*(\bar{k}) & Y_3(\bar{k}) \end{bmatrix} \quad \text{and}$$

$$\mathrm{in}_-\left(\begin{bmatrix} X(\bar{k}) & X_2(\bar{k}) \\ X_2^*(\bar{k}) & X_3(\bar{k}) \end{bmatrix}\right) = q_- \quad \text{for all } k \in \mathcal{C}_i^{(j)}. \quad (19)$$

We can repeat this procedure for all the other values of $t \in \mathbb{Z}$ and $\mathcal{C}_i^{(j)} \in \mathcal{C}^{(j)}$, and then following the same argument as that at the end of the proof of Lemma 12, we can show that these matrix sequences can be chosen to be uniformly bounded from below and above. This ensures that the graph-diagonal operators specified from these sequences would satisfy the desired invertibility and boundedness conditions. Finally, (19) ensures equal negative inertia over each permutation cycle, and hence condition (15) is also met. ■

Theorem 14: Given G as in (1), a vector-valued sequence $(n_{K0},\ldots,n_{Ks})$, and $(n_0,\ldots,n_s) := p(A)$, then there exists an admissible synthesis K for G with realization dimensions satisfying $p(A_K) \leq (n_{K1},\ldots,n_{Kd})$ if there exist X and Y in $\mathcal{X}$ such that, for each $j \in \{1,\ldots,s\}$, $\mathcal{C}_i^{(j)} \in \mathcal{C}^{(j)}$ and $t \in \mathbb{Z}$, the inequality $n_j(t,k)+n_{Kj}(t,k) \geq$

$$\mathrm{In}_+\left(\begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix}\right)(t,k) + \overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix}\right)(t,\mathcal{C}_i^{(j)})$$

holds for all $k \in \mathcal{C}_i^{(j)}$, and furthermore,

$$\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, S^*XS, X, N_X\right) \prec 0, \quad (20)$$

$$\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^*, Y, S^*YS, N_Y\right) \prec 0, \quad (21)$$

$$\begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} \succeq 0, \quad \mathrm{In}_+\left(\begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix}\right) \leq n_0 + n_{K0},$$

where N_X and N_Y are coercive operators with $\mathrm{Im}\, N_X = \mathrm{Ker}\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ and $\mathrm{Im}\, N_Y = \mathrm{Ker}\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}$.

Proof: The inertia conditions, together with the positive semidefinite inequality on X_0 and Y_0, ensure by lemmas 12 and 13 the existence of a partitioned graph-diagonal operator $X_{cl} \in \mathcal{X}_{cl}$ satisfying $\mathrm{In}_-(X_{cl}) = \mathrm{In}_-(S_{cl}^* X_{cl} S_{cl})$ and (10). Then, invoking Proposition 11, the inequalities (20) and (21) are equivalent to (8), which, by Lemma 10, implies the existence of the desired synthesis. ■

The inertia conditions in the previous theorem are not convex, however, they can be trivialized by assuming controllers of sufficiently large dimensions. For $j \in \{1,\ldots,s\}$, define

$$n_j^{\max}(t,k) := \max_{k \in V} n_j(t,k), \quad (22)$$

Corollary 15: Given a nominal system G as in (1), suppose there exist X and Y in $\mathcal{X}$ satisfying (20), (21) and

$$\begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} \succeq 0.$$

Then there exists an admissible controller K with realization dimensions satisfying

$$p(A_K) \leq (n_0, n_1 + 2n_1^{\max}, n_2 + 2n_2^{\max}, \ldots, n_s + 2n_s^{\max}),$$

where $(n_0,\ldots,n_s) := p(A)$ and $n_j^{\max}$ is defined in (22).

This corollary gives sufficient convex conditions for the existence of an admissible synthesis. The solutions can then be used to construct a controller; this is formally identical to the construction in [2] except the pointwise indexing is now over the vertices of the graph.

VI. Conclusions

This paper introduces a framework in which to consider distributed control over infinite graphs, and provides analysis and convex synthesis conditions for design in this setting.

References

[1] R. D'Andrea and G.E. Dullerud, "Control of spatially distributed systems," *IEEE TAC*, 48:1478–1495, 2003.

[2] G.E. Dullerud, R. D'Andrea, "Distributed control of heterogeneous systems," *IEEE TAC*, 49:2113–2128, 2004.

[3] A. Ben-Artzi, and I. Gohberg, "Inertia theorems for nonstationary discrete systems and dichotomy", *Linear Alg. Appl.*, 120:95–138, 1989.

[4] M. Farhood and G. E. Dullerud, "LMI tools for eventually periodic systems," *Syst. Cont. L.*, 47:417–432, 2002.

[5] A. Packard and J. C. Doyle, "The complex structured singular value," *Automatica*, 29:71–109, 1993.

[6] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *Int. J. Robust Nonlinear Control*, 4:421–448, 1991.

[7] C. Langbort, R.S. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph", *IEEE TAC*, 49:1502–1519, 2004.

[8] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Control Lett.*, 22:79–92, 1994.

[9] B. Recht, R. D'Andrea, "Distributed control of systems over discrete groups", *IEEE TAC*, 49:1446–1452, 2004.