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Brief paper

Control of nonstationary LPV systems☆

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Abstract

This paper considers control of nonstationary linear parameter-varying systems, and is motivated by interest in the control of nonlinear systems along prespecified trajectories. In the paper, synthesis conditions are derived for such systems using an operator theoretical framework with the ℓ_2 induced norm as the performance measure. These conditions are given in terms of structured operator inequalities. In general, evaluating the validity of these conditions is an infinite dimensional convex optimization problem; however, if the initial system is *eventually periodic*, they reduce to a finite dimensional semi-definite programming problem. The paper concludes with an in-depth example on the control of a two-thruster hovercraft along an eventually periodic trajectory.

Keywords: Linear parameter varying systems; Time-varying systems; LFT; Periodic orbits; LMIs

1. Introduction

This paper deals with the generalization of results for linear parameter-varying (LPV) systems to the nonstationary (time-varying) case. Our work is motivated by the desire to control nonlinear systems along prespecified trajectories, and the results obtained are computable as finite dimensional convex programs when the trajectories involved are *eventually periodic*. That is to say that the trajectory can be arbitrary for a finite amount of time, but then settles into a periodic orbit; a special case of this is when a system transitions between two operating points.

The types of plant models we consider are of the form

$$\begin{aligned} x(k+1) &= A(\delta(k), k)x(k) + B(\delta(k), k)u(k), \\ y(k) &= C(\delta(k), k)x(k) + D(\delta(k), k)u(k), \end{aligned} \qquad (1)$$

where $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, $C(\cdot,\cdot)$, and $D(\cdot,\cdot)$ are matrix-valued functions that are known *a priori*. The variable k is time, and $\delta(k) := (\delta_1(k), \ldots, \delta_d(k))$ is a vector of scalar parameters that *are not known a priori*. We will be concerned with the situation where both of the following conditions hold: (i) the parameters $\delta_i(k)$, although not known a priori, are available for measurement at time k; and (ii) A, B, C, and D are rational functions of the parameters δ_i at each instant k. Such models arise naturally when, for instance, expanding a nonlinear system around a trajectory, and the scenario satisfying (i) and (ii) represents a type of gain scheduling.

The main idea for the use of this type of model originates in the seminal papers Lu, Zhou, and Doyle (1996) and Packard (1994), where gain scheduling based on linear fractional transformation (LFT) models is introduced. These papers consider the case of stationary LPV systems, and one of the main contributions of the current paper is the generalization of the results in Packard (1994) to the general nonstationary case. This is accomplished by combining the approach taken in Packard (1994) for the stationary case with the framework developed in Dullerud and Lall (1999) for linear time-varying systems. In addition to the approach taken in Packard (1994), the proof technique in this paper parallels that in Gahinet and Apkarian (1991), which considers the time-invariant H_∞ problem. Other closely related works on stationary LPV models and on nonstationary systems appear in Apkarian and Gahinet (1995), Ball, Gohberg, and Kaashoek (1992), Halanay

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and Ionescu (1994), Helmersson (1995), Iglesias (1996), Lee (1997), Lu et al. (1996), Wu (2001) and Wu, Packard, and Becker (1996) respectively.

We remark that in many cases, when deriving a nonstationary linear parameter-varying (NSLPV) model such as (1) along a trajectory, it is possible to find a *stationary* LPV model that also parameterizes the nonlinear system along the trajectory. The main advantages that a nonstationary model will typically have are: (a) from an algebraic point of view, one can easily construct situations in which the stationary LPV system is not stabilizable, but the nonstationary one is; and (b) the set of systems parameterized by the stationary model will typically be much larger than that by the corresponding nonstationary one, and thus may needlessly limit the closed-loop performance of the model. Indeed, the situation in (a) can be viewed as an extreme version of type (b) conservatism. In time-varying systems, it is well known that systems can be stabilizable even though the state space matrices, pointwise in time, are not stabilizable; this serves as an analogy for (a) in the case of standard systems.

The main contributions of this paper are:

- The development of general synthesis conditions for control of NSLPV systems; these conditions are infinite dimensional and convex. As with the stationary case, the conditions we obtain are only sufficient for an LPV synthesis to exist, but are necessary and sufficient for the case where there are no parameters δ_i; that is, the nominal system is a standard time-varying system. When the model in (1) is stationary, the conditions derived are exactly the ones in Packard (1994).
- The introduction of the concept of an *eventually periodic* LPV system, and results showing that, for these systems, the general synthesis conditions obtained become linear matrix inequalities (LMIs), thus making them readily computable. Eventually periodic systems contain both finite horizon and periodic systems as special cases. In addition to the application already mentioned of trajectories that eventually settle into a periodic orbit, eventually periodic systems naturally arise when considering problems in which the plant has an uncertain initial state.

The paper is based on Farhood (2005) and is organized as follows. In Section 2, we define our notation and introduce some useful machinery. We formulate the LPV problem of interest in Section 3, and develop analysis and synthesis results in Section 4. In Section 5, we consider eventually periodic LPV systems, and we conclude in Section 6 with an example on the control of a two-thruster hovercraft along an eventually periodic path.

2. Preliminaries

The set of real $n \times m$ matrices is denoted by $\mathbb{R}^{n\times m}$. The linear space of elements $x = (x(0), x(1), x(2), \ldots)$, with $x(k) \in \mathbb{R}^{n(k)}$, is denoted by $\ell(\mathbb{R}^n)$. We define the Hilbert space $\ell_2(\mathbb{R}^n)$ as the subspace of $\ell(\mathbb{R}^n)$ consisting of elements $x \in \ell(\mathbb{R}^n)$ such that $\|x\|^2 = \sum_{k=0}^{\infty} x(k)^* x(k) < \infty$. When the spatial dimensions $n(k)$ are either evident or irrelevant to the discussion, we will simply use the abbreviations ℓ_2 and ℓ. We denote the space of bounded linear operators mapping ℓ_2 to ℓ_2 by $\mathcal{L}(\ell_2)$, and the ℓ_2 to ℓ_2 induced norm of an operator X by $\|X\|$. The adjoint of X is written X^*. When an operator $X \in \mathcal{L}(\ell_2)$ is self-adjoint, we use $X \prec 0$ to mean it is negative definite; that is there exists a number $\alpha > 0$ such that, for all nonzero $x \in \ell_2$, the inequality $\langle x, Xx\rangle < -\alpha\|x\|^2$ holds, where $\langle\cdot,\cdot\rangle$ denotes the inner product. Given a sequence of dimensions $n_1(k), n_2(k), \ldots, n_p(k)$, we define the Hilbert space direct sum $\ell_2^{(n_1,\ldots,n_p)} := \ell_2(\mathbb{R}^{n_1}) \oplus \ell_2(\mathbb{R}^{n_2}) \oplus \cdots \oplus \ell_2(\mathbb{R}^{n_p})$. Let $0_{i\times j}$ and I_j denote an $i \times j$ zero matrix and a $j \times j$ identity matrix respectively. Other notations used in this paper are

$$I_{\ell_2}^n := \operatorname{diag}(I_{n(0)}, I_{n(1)}, I_{n(2)}, \ldots),$$
$$I_{\ell_2}^{(n_1,\ldots,n_p)} := \operatorname{diag}(I_{\ell_2}^{n_1}, I_{\ell_2}^{n_2}, \ldots, I_{\ell_2}^{n_p}),$$
$$0_{\ell_2}^{n\times m} := \operatorname{diag}(0_{n(0)\times m(0)}, 0_{n(1)\times m(1)}, \ldots),$$

and

$$0_{\ell_2}^{(n_1,\ldots,n_p)\times(m_1,\ldots,m_q)} := \begin{bmatrix} 0_{\ell_2}^{1\times n_1} & \cdots & 0_{\ell_2}^{1\times n_p}\end{bmatrix}^* \times \begin{bmatrix} 0_{\ell_2}^{1\times m_1} & \cdots & 0_{\ell_2}^{1\times m_q}\end{bmatrix}.$$

A key operator used in the paper is the unilateral shift Z, defined as follows:

$$Z:\ \ell_2\left(\mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots\right) \to \ell_2\left(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots\right)$$
$$(a(1), a(2), \ldots) \xmapsto{Z} (0, a(1), a(2), \ldots).$$

Clearly this definition is extendable to ℓ, and in the sequel, we will not distinguish between these mappings.

Following the notation and approach in Dullerud and Lall (1999), we make the following definitions. First, we say a linear operator Q mapping $\ell(\mathbb{R}^{m(0)}, \mathbb{R}^{m(1)}, \ldots)$ to $\ell(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \ldots)$ is *block-diagonal* if there exists a sequence of matrices $Q(k)$ in $\mathbb{R}^{n(k)\times m(k)}$ such that, for all w, z, if $z = Qw$, then $z(k) = Q(k)w(k)$. Then Q has the representation $\operatorname{diag}(Q(0), Q(1), Q(2), \ldots)$.

Suppose F, G, R and S are block-diagonal operators, and let A be a *partitioned* operator of the form $A = \begin{bmatrix} F & G \\ R & S\end{bmatrix}$. Then we define

$$[\![A]\!] := \operatorname{diag}\left(\begin{bmatrix} F(0) & G(0) \\ R(0) & S(0)\end{bmatrix}, \begin{bmatrix} F(1) & G(1) \\ R(1) & S(1)\end{bmatrix}, \ldots\right).$$

Clearly, $[\![A]\!]$ is simply A with the rows and columns permuted appropriately so that $[\![A]\!]_k = \begin{bmatrix} F(k) & G(k) \\ R(k) & S(k)\end{bmatrix}$. It is easy to see that $[\![A+B]\!] = [\![A]\!] + [\![B]\!]$ and $[\![AC]\!] = [\![A]\!][\![C]\!]$ hold for appropriately dimensioned operators, and that $A \prec \beta I$ holds if and only if $[\![A]\!] \prec \beta I$, where β is a scalar. Namely, the $[\![\bullet]\!]$ operation is a homomorphism from partitioned operators with block-diagonal entries to block-diagonal operators.

3. Problem formulation

We will be concerned with a particular subclass of models of the form in (1), where the dependence of the state-space matrices on the parameters δ_i is given in terms of a feedback coupling. These types of systems are commonly referred to as LFT systems, in which the state-space dependence on the parameters is rational. Models of this subclass are the straightforward generalization of the LPV systems first introduced in Lu et al. (1996) and Packard (1994).

3.1. NSLPV plant

Let G_δ be a discrete-time LFT system defined by the following state-space equations:

$$\begin{bmatrix} x(k+1) \\ \alpha(k) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A_{ss}(k) & A_{sp}(k) & B_{1s}(k) & B_{2s}(k) \\ A_{ps}(k) & A_{pp}(k) & B_{1p}(k) & B_{2p}(k) \\ C_{1s}(k) & C_{1p}(k) & D_{11}(k) & D_{12}(k) \\ C_{2s}(k) & C_{2p}(k) & D_{21}(k) & 0 \end{bmatrix} \times \begin{bmatrix} x(k) \\ \beta(k) \\ w(k) \\ u(k) \end{bmatrix}, \tag{2}$$

$$\beta(k) = \mathrm{diag}(\delta_1(k) I_{n_1(k)}, \ldots, \delta_d(k) I_{n_d(k)})\alpha(k) = \underline{\Delta}(k)\alpha(k),$$

$x(0) = 0$, for $w \in \ell_2$. The signals $w(k)$ and $z(k)$ denote the exogenous disturbances and errors, respectively, whereas $u(k)$ denotes the applied control and $y(k)$ the measurements. The vectors $x(k), \alpha(k), \beta(k), z(k), w(k), y(k)$, and $u(k)$ are real and have time-varying dimensions, denoted by $n_0(k)$, $n(k)$, $n(k)$, $n_z(k)$, $n_w(k)$, $n_y(k)$, and $n_u(k)$ respectively. The parameters $\delta_i(k)$ are real scalars such that $|\delta_i(k)| \le 1$ for all $k \ge 0$ and $i = 1, 2, \ldots, d$, and the associated dimensions $n_i(k)$ satisfy $\sum_{i=1}^{d} n_i(k) = n(k)$. We assume that $I - A_{pp}(k)\underline{\Delta}(k)$ is invertible for all $k \ge 0$ so that this LFT system is well posed, and thus there are unique solutions in ℓ to (2). Also, we assume all the state-space matrices are uniformly bounded functions of time.

Using the previously defined notation, clearly the matrix sequences $A_{ss}(k)$, $B_{1s}(k)$, $B_{2s}(k)$, $C_{1s}(k)$, $C_{2s}(k)$, $D_{11}(k)$, $D_{12}(k)$, and $D_{21}(k)$ define bounded block-diagonal operators. The blocks of matrix $\underline{\Delta}(k)$ naturally partition $\alpha(k)$ and $\beta(k)$ into d separate vector-valued channels, conformably with which we partition the following state-space matrices:

$$\begin{aligned} A_{sp}(k) &= \begin{bmatrix} A_{sp}^1(k) & A_{sp}^2(k) & \cdots & A_{sp}^d(k) \end{bmatrix} \\ C_{1p}(k) &= \begin{bmatrix} C_{1p}^1(k) & C_{1p}^2(k) & \cdots & C_{1p}^d(k) \end{bmatrix} \\ C_{2p}(k) &= \begin{bmatrix} C_{2p}^1(k) & C_{2p}^2(k) & \cdots & C_{2p}^d(k) \end{bmatrix} \\ A_{pp}(k) &= \begin{bmatrix} A_{pp}^{11}(k) & \cdots & A_{pp}^{1d}(k) \\ \vdots & \ddots & \vdots \\ A_{pp}^{d1}(k) & \cdots & A_{pp}^{dd}(k) \end{bmatrix} \\ A_{ps}(k) &= \begin{bmatrix} A_{ps}^1(k) \\ A_{ps}^2(k) \\ \vdots \\ A_{ps}^d(k) \end{bmatrix} \end{aligned} \tag{3}$$

$$B_{1p}(k) = \begin{bmatrix} B_{1p}^1(k) \\ B_{1p}^2(k) \\ \vdots \\ B_{1p}^d(k) \end{bmatrix} \qquad B_{2p}(k) = \begin{bmatrix} B_{2p}^1(k) \\ B_{2p}^2(k) \\ \vdots \\ B_{2p}^d(k) \end{bmatrix},$$

where $A_{sp}^i(k) \in \mathbb{R}^{n_0(k+1)\times n_i(k)}$, $A_{pp}^{ij}(k) \in \mathbb{R}^{n_i(k)\times n_j(k)}$, $A_{ps}^i(k) \in \mathbb{R}^{n_i(k)\times n_0(k)}$, $B_{1p}^i(k) \in \mathbb{R}^{n_i(k)\times n_w(k)}$, $B_{2p}^i(k) \in \mathbb{R}^{n_i(k)\times n_u(k)}$, $C_{1p}^i(k) \in \mathbb{R}^{n_z(k)\times n_i(k)}$, and $C_{2p}^i(k) \in \mathbb{R}^{n_y(k)\times n_i(k)}$. The matrix sequence of each of the elements of the state-space matrices in (3) defines a bounded block-diagonal operator; and so we construct from the sequence of each of these state-space matrices a partitioned operator, each of whose elements is block diagonal and defined in the obvious way. For instance, the matrix sequences $A_{sp}^1(k), \ldots, A_{sp}^d(k)$ define block-diagonal operators that compose the partitioned operator A_{sp}. With Z being the shift, we can rewrite our system equations as

$$\begin{bmatrix} x \\ \alpha \\ z \\ y \end{bmatrix} = \begin{bmatrix} ZA_{ss} & ZA_{sp} & ZB_{1s} & ZB_{2s} \\ A_{ps} & A_{pp} & B_{1p} & B_{2p} \\ C_{1s} & C_{1p} & D_{11} & D_{12} \\ C_{2s} & C_{2p} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ \beta \\ w \\ u \end{bmatrix}, \qquad \begin{bmatrix} x \\ \beta \end{bmatrix} = \mathrm{diag}(I_{\ell_2}^{n_0}, \Delta_1, \ldots, \Delta_d) \begin{bmatrix} x \\ \alpha \end{bmatrix} = \Delta \begin{bmatrix} x \\ \alpha \end{bmatrix}, \tag{4}$$

where $x \in \ell(\mathbb{R}^{n_0})$, $w \in \ell_2(\mathbb{R}^{n_w})$, $q \in \ell(\mathbb{R}^{n_q})$ for $q = u, z, y$, $\beta = (\beta_1, \ldots, \beta_d)$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta_i, \alpha_i \in \ell(\mathbb{R}^{n_i})$, and $\Delta_i = \mathrm{diag}(\delta_i(0) I_{n_i(0)}, \delta_i(1) I_{n_i(1)}, \delta_i(2) I_{n_i(2)}, \ldots)$.

We now introduce some convenient definitions and notations. To start, we define

$$A := \begin{bmatrix} A_{ss} & A_{sp} \\ A_{ps} & A_{pp} \end{bmatrix}, \qquad B_1 := \begin{bmatrix} B_{1s} \\ B_{1p} \end{bmatrix}, \qquad B_2 := \begin{bmatrix} B_{2s} \\ B_{2p} \end{bmatrix},$$
$$C_1 := \begin{bmatrix} C_{1s} & C_{1p} \end{bmatrix}, \qquad C_2 := \begin{bmatrix} C_{2s} & C_{2p} \end{bmatrix}.$$

We also define $\widetilde{Z} = \mathrm{diag}(Z, I)$, which is partitioned similarly to A. This partitioning is clearly conformable to that of $\Delta = \mathrm{diag}(\Delta_s, \Delta_p)$, where $\Delta_s = I_{\ell_2}$ and $\Delta_p = \mathrm{diag}(\Delta_1, \ldots, \Delta_d)$. Notice that $[\![\Delta_p]\!]_k = \underline{\Delta}(k)$. Moreover, we define $\mathbf{\Delta} := \{\Delta \in \mathcal{L}(\ell_2^{(n_0,\ldots,n_d)}) : \Delta \text{ is partitioned as in (4)}, \|\Delta\| \le 1\}$. The set of systems G_δ for all $\Delta \in \mathbf{\Delta}$ defines an NSLPV model $\mathcal{G}_\delta$, namely $\mathcal{G}_\delta = \{G_\delta : \Delta \in \mathbf{\Delta}\}$.

We now define the basic notions of well-posedness and stability for NSLPV models.

Definition 1. An NSLPV model $\mathcal{G}_\delta$ is

(i) well posed if $I - \Delta_p A_{pp}$ has an algebraic inverse (not necessarily bounded) for all $\Delta \in \mathbf{\Delta}$;
(ii) ℓ_2-stable if $I - \Delta\widetilde{Z}A$ has a bounded inverse for all $\Delta \in \mathbf{\Delta}$.

It follows from (ii) that, when a system is ℓ_2-stable, there exists a unique $(x, \beta) \in \ell_2^{(n_0,\ldots,n_d)}$ satisfying (4). We refer the reader to Farhood and Dullerud (2007) for an in-depth treatment of well-posedness and stability for NSLPV systems.

The parameters δ_i are not uncertain, and while not known a priori, they are available for measurement at each k. Next we incorporate these plant parameters into the control design.

3.2. Controller and closed-loop system

Given an operator $\Delta \in \mathbf{\Delta}$ and an associated system G_δ, we suppose this system is being controlled by a controller K_δ that has a similar structure as G_δ. The controller is defined by the state-space equations

$$\begin{bmatrix} x^K \\ \alpha^K \\ u \end{bmatrix} = \begin{bmatrix} ZA^K_{ss} & ZA^K_{sp} & ZB^K_s \\ A^K_{ps} & A^K_{pp} & B^K_p \\ C^K_s & C^K_p & D^K \end{bmatrix} \begin{bmatrix} x^K \\ \beta^K \\ y \end{bmatrix}, \qquad (5)$$

$$\begin{bmatrix} x^K \\ \beta^K \end{bmatrix} = \mathrm{diag}(I^{r_0}_{\ell_2}, \Delta^K_1, \ldots, \Delta^K_d) \begin{bmatrix} x^K \\ \alpha^K \end{bmatrix} = \Delta^K \begin{bmatrix} x^K \\ \alpha^K \end{bmatrix},$$

where $x^K \in \ell(\mathbb{R}^{m_0})$, $\beta^K = (\beta^K_1, \ldots, \beta^K_d)$, $\alpha^K = (\alpha^K_1, \ldots, \alpha^K_d)$, β^K_i, $\alpha^K_i \in \ell(\mathbb{R}^{m_i})$, and the block-diagonal operator $\Delta^K_i = \mathrm{diag}(\delta_i(0) I_{m_i(0)}, \delta_i(1) I_{m_i(1)}, \ldots)$. We will derive the proper values for the controller dimensions in the next section. Note that the parameters δ that affect the controller are the same as those that affect the plant. The well-posedness of this LFT can always be guaranteed by slightly perturbing Δ^K, if necessary, to ensure that $I - A^K_{pp}\Delta^K$ is invertible.

We write the realization of the closed-loop system as

$$x_{cl} = \Delta_{cl} \widetilde{Z}_2 A_{cl} x_{cl} + \Delta_{cl} \widetilde{Z}_2 B_{cl} w, \qquad z = C_{cl} x_{cl} + D_{cl} w,$$

where x_{cl} is the column vector (x, β, x^K, β^K), $\Delta_{cl} = \mathrm{diag}(\Delta, \Delta^K)$, $\widetilde{Z}_2 = \mathrm{diag}(\widetilde{Z}, \widetilde{Z})$, and the remaining operators are defined in the obvious way. We denote this realization by S_δ, and hence the closed-loop NSLPV system $\mathcal{S}_\delta$ is given by $\mathcal{S}_\delta = \{S_\delta : \Delta \in \mathbf{\Delta}\}$. Note that, for NSLPV systems, the standard form of Δ is the one given in (4) and (5). It is obvious that Δ_{cl} is not of the standard form, but this can be remedied easily by a change of basis via a permutation. Specifically, there exists a unique permutation P such that

$$P^* \Delta_{cl} P = \Delta^L = \mathrm{diag}(I^{s_0}_{\ell_2}, \Delta^L_1, \ldots, \Delta^L_d), \qquad (6)$$

where $s_i = n_i + m_i$, $\Delta^L_i = \mathrm{diag}(\delta_i(0) I_{s_i(0)}, \delta_i(1) I_{s_i(1)}, \ldots)$; clearly, Δ^L conforms with the standard form. Then, for all $\Delta \in \mathbf{\Delta}$, an equivalent realization for S_δ is given by

$$\begin{bmatrix} \begin{bmatrix} x^L \\ \alpha^L \end{bmatrix} \\ z \end{bmatrix} = \begin{bmatrix} \widetilde{Z} A^L & \widetilde{Z} B^L \\ C^L & D^L \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x^L \\ \beta^L \end{bmatrix} \\ w \end{bmatrix},$$

$$\begin{bmatrix} x^L \\ \beta^L \end{bmatrix} = \Delta^L \begin{bmatrix} x^L \\ \alpha^L \end{bmatrix},$$

where $x^L \in \ell(\mathbb{R}^{s_0})$, $\alpha^L = (\alpha^L_1, \ldots, \alpha^L_d)$, $\beta^L = (\beta^L_1, \ldots, \beta^L_d)$, $\alpha^L_i, \beta^L_i \in \ell(\mathbb{R}^{s_i})$, and $A^L = (\widetilde{Z}^* P^* \widetilde{Z}_2) A_{cl} P$,

$$B^L = (\widetilde{Z}^* P^* \widetilde{Z}_2) B_{cl}, \qquad C^L = C_{cl} P, \qquad D^L = D_{cl}. \qquad (7)$$

Notice that $\widetilde{Z} A^L = \widetilde{Z}\widetilde{Z}^*(P^* \widetilde{Z}_2 A_{cl} P) = P^* \widetilde{Z}_2 A_{cl} P$ and $\widetilde{Z} B^L = P^* \widetilde{Z}_2 B_{cl}$. For convenience, we define $\mathbf{\Delta}^L := \{\Delta^L \in \mathcal{L}(\ell_2^{(s_0,\ldots,s_d)}) : \Delta^L \text{ is partitioned as in (6)}, \left\| \Delta^L \right\| \le 1\}$.

We will denote the NSLPV controller by K_δ, instead of $\mathcal{K}_\delta$, to emphasize that the controller parameters are not arbitrary but rather they are the same as those of the plant.

4. Synthesis

The following definition expresses our synthesis goal.

Definition 2. A controller K_δ is an *admissible synthesis* for an NSLPV plant $\mathcal{G}_\delta$ if the closed-loop NSLPV system $\mathcal{S}_\delta$ is ℓ_2-stable and the input-output mapping $w \mapsto z$ satisfies

$$\|w \mapsto z\| = \left\| C^L (I - \Delta^L \widetilde{Z} A^L)^{-1} \Delta^L \widetilde{Z} B^L + D^L \right\| < 1$$

for all $\Delta^L \in \mathbf{\Delta}^L$.

Lemma 3. *The closed-loop NSLPV system $\mathcal{S}_\delta$ is ℓ_2-stable and the performance inequality $\|w \mapsto z\| < 1$ is satisfied for all $\Delta \in \mathbf{\Delta}$ if there exists a positive definite operator $\bar{X}^L$ in the commutant of $\mathbf{\Delta}^L$ such that*

$$\begin{bmatrix} \widetilde{Z} A^L & \widetilde{Z} B^L \\ C^L & D^L \end{bmatrix}^* \begin{bmatrix} \bar{X}^L & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{Z} A^L & \widetilde{Z} B^L \\ C^L & D^L \end{bmatrix} - \begin{bmatrix} \bar{X}^L & 0 \\ 0 & I \end{bmatrix} \prec 0. \qquad (8)$$

This is a generalization of the sufficiency part of the Kalman-Yakubovich-Popov Lemma. Its proof is routine and so is omitted. Note that inequality (8) is necessary and sufficient in the purely time-varying case (Yacubovich, 1975); but, in our case, it is in general only sufficient, and thus this may introduce conservatism into our approach.

At this point, we define the set $\mathcal{X}^L$ to consist of positive definite operators X^L of the form

$$X^L = \mathrm{diag}(X^L_0, X^L_1, \ldots, X^L_d) \succ 0,$$

where each X^L_i is block diagonal, namely

$$X^L_i = \mathrm{diag}(X^L_i(0), X^L_i(1), \ldots), \quad \text{with } X^L_i(k) \in \mathbb{R}^{s_i(k) \times s_i(k)}.$$

Proposition 4. *A positive definite solution $\bar{X}^L$, belonging to the commutant of $\mathbf{\Delta}^L$, exists to inequality (8) if and only if a solution $X^L \in \mathcal{X}^L$ exists.*

Proof. The proof of the "if" direction is immediate since $\mathcal{X}^L$ is clearly a subset of the commutant of $\mathbf{\Delta}^L$.

We now prove the "only if" direction. Suppose $\bar{X}^L$ is a positive definite operator in the commutant of $\mathbf{\Delta}^L$ satisfying (8). Then $\bar{X}^L$ has the form $\bar{X}^L = \mathrm{diag}(\bar{X}^L_0, \bar{X}^L_1, \ldots, \bar{X}^L_d) \succ 0$, where, for $i = 1, 2, \ldots, d$, each operator $\bar{X}^L_i$ is block diagonal with $[\![\bar{X}^L_i]\!]_k = \bar{X}^L_i(k) \in \mathbb{R}^{s_i(k) \times s_i(k)}$, whereas $\bar{X}^L_0 \in \mathcal{L}(\ell_2(\mathbb{R}^{s_0}))$ and is not necessarily block diagonal. Following a very similar argument to that used in the proof of Dullerud and Lall (1999, Theorem 11), we can construct from $\bar{X}^L$ an operator $X^L = \mathrm{diag}(X^L_0, X^L_1, \ldots, X^L_d) \in \mathcal{X}^L$ that also solves (8) such that X^L_0 is a block-diagonal operator whose elements are the blocks on the diagonal of $\bar{X}^L_0$ and $X^L_i = \bar{X}^L_i$ for $i = 1, \ldots, d$. ■

The procedure henceforth is very similar to the ones given in Dullerud and Lall (1999) and Gahinet and Apkarian (1991) for LTI and LTV systems respectively, and so we present it very briefly. Before proceeding, it is convenient to define the notations $\bar{n} = (n_0, \ldots, n_d)$ and $\bar{m} = (m_0, \ldots, m_d)$. Now, consider the following closed-loop parametrization:

$$\begin{aligned} A_{cl} &= \bar{A} + \underline{B} J \underline{C}, \qquad B_{cl} = \bar{B} + \underline{B} J \underline{D}_{21}, \\ C_{cl} &= \bar{C} + \underline{D}_{12} J \underline{C}, \qquad D_{cl} = D_{11} + \underline{D}_{12} J \underline{D}_{21}, \end{aligned} \tag{9}$$

where operator J describes the controller realization, namely

$$J := \begin{bmatrix} A_{ss}^K & A_{sp}^K & B_s^K \\ A_{ps}^K & A_{pp}^K & B_p^K \\ C_s^K & C_p^K & D^K \end{bmatrix},$$

and the other operators are defined as follows:

$$\bar{C} := \begin{bmatrix} C_1 & 0_{\ell_2}^{n_z \times \bar{m}} \end{bmatrix}, \qquad \underline{D}_{12} := \begin{bmatrix} 0_{\ell_2}^{n_z \times \bar{m}} & D_{12} \end{bmatrix},$$

$$\bar{A} := \begin{bmatrix} A & 0 \\ 0 & 0_{\ell_2}^{\bar{m} \times \bar{m}} \end{bmatrix}, \qquad \bar{B} := \begin{bmatrix} B_1 \\ 0_{\ell_2}^{\bar{m} \times n_w} \end{bmatrix},$$

$$\underline{B} := \begin{bmatrix} 0 & B_2 \\ I_{\ell_2}^{\bar{m}} & 0 \end{bmatrix},$$

$$\underline{C} := \begin{bmatrix} 0 & I_{\ell_2}^{\bar{m}} \\ C_2 & 0 \end{bmatrix}, \qquad \underline{D}_{21} := \begin{bmatrix} 0_{\ell_2}^{\bar{m} \times n_w} \\ D_{21} \end{bmatrix}.$$

Lemma 5. *The controller K_δ described by the operator J is an admissible synthesis if there exists $X^L \in \mathcal{X}^L$ such that*

$$H_{X_P^L} + Q^* J^* R + R^* J Q \prec 0, \tag{10}$$

where $R = \begin{bmatrix} \underline{B}^* & 0_{\ell_2}^{(\bar{m}, n_u) \times (\bar{n}, \bar{m})} & 0_{\ell_2}^{(\bar{m}, n_u) \times n_w} & \underline{D}_{12}^* \end{bmatrix}$,

$$Q = \begin{bmatrix} 0_{\ell_2}^{(\bar{m}, n_y) \times (\bar{n}, \bar{m})} & \underline{C} & \underline{D}_{21} & 0_{\ell_2}^{(\bar{m}, n_y) \times n_z} \end{bmatrix},$$

$$H_{X_P^L} = \begin{bmatrix} -\widetilde{Z}_2^* (X_P^L)^{-1} \widetilde{Z}_2 & \bar{A} & \bar{B} & 0 \\ \bar{A}^* & -X_P^L & 0 & \bar{C}^* \\ \bar{B}^* & 0 & -I & D_{11}^* \\ 0 & \bar{C} & D_{11} & -I \end{bmatrix},$$

$$X_P^L = P X^L P^*,$$

and the permutation P is as defined in (6).

Proof. The controller is admissible if there exists a solution $X^L \in \mathcal{X}^L$ to inequality (8). Pre- and post-multiplying (8) by $\mathrm{diag}(P, I)$ and $\mathrm{diag}(P^*, I)$ respectively and then substituting the definitions from (7), we get the equivalent inequality

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}^* \begin{bmatrix} \widetilde{Z}_2^* P X^L P^* \widetilde{Z}_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} - \begin{bmatrix} P X^L P^* & 0 \\ 0 & I \end{bmatrix} \prec 0.$$

Applying the Schur complement formula twice and then substituting expressions (9) give the desired result. ■

Lemma 6. *There exists a partitioned operator J satisfying inequality (10) if and only if*

$$W_R^* H_{X_P^L} W_R \prec 0 \quad \text{and} \quad W_Q^* H_{X_P^L} W_Q \prec 0, \tag{11}$$

where $\mathrm{Im}\, W_R = \mathrm{Ker}\, R$, $\mathrm{Im}\, W_Q = \mathrm{Ker}\, Q$, $W_R^* W_R = I$, and $W_Q^* W_Q = I$.

This lemma is a generalization of a similar result in Gahinet and Apkarian (1991). Its proof is nearly identical to the one presented in Dullerud and Lall (1999) and so we omit it.

One problem with the preceding result is that inequalities (11) are not affine in X_P^L, since both X_P^L and $(X_P^L)^{-1}$ appear in $H_{X_P^L}$. To remedy this, given $X^L = \mathrm{diag}(X_0^L, \ldots, X_d^L) \in \mathcal{X}^L$, we conveniently partition the matrix blocks as

$$X_i^L(k) = \begin{bmatrix} X_i(k) & X_i^b(k) \\ X_i^b(k)^* & X_i^c(k) \end{bmatrix},$$

where the matrices $X_i(k) \in \mathbb{R}^{n_i(k) \times n_i(k)}$, $X_i^b(k) \in \mathbb{R}^{n_i(k) \times m_i(k)}$ and $X_i^c(k) \in \mathbb{R}^{m_i(k) \times m_i(k)}$. Note that the sequences $X_i(k)$, $X_i^b(k)$, and $X_i^c(k)$ define block-diagonal operators X_i, X_i^b and X_i^c respectively. With $X_P^L = P X^L P^*$, it is straightforward to verify that

$$X_P^L = \begin{bmatrix} X & X^b \\ (X^b)^* & X^c \end{bmatrix},$$

where $X = \mathrm{diag}(X_0, \ldots, X_d)$, and X^b and X^c are defined similarly. Clearly, since $(X_P^L)^{-1} = P (X^L)^{-1} P^*$, then $(X_P^L)^{-1}$ has the same form as X_P^L, namely

$$X_P^L = \begin{bmatrix} X & X^b \\ (X^b)^* & X^c \end{bmatrix}, \qquad (X_P^L)^{-1} = \begin{bmatrix} Y & Y^b \\ (Y^b)^* & Y^c \end{bmatrix}. \tag{12}$$

We define the set $\mathcal{X}$ to consist of positive definite operators $X = \mathrm{diag}(X_0, \ldots, X_d)$, where each $X_i \in \mathcal{L}(\ell_2^{n_i})$ is block diagonal. Then, X and Y from (12) are elements of $\mathcal{X}$.

Lemma 7. *Suppose $X, Y \in \mathcal{X}$ and m_i is a positive integer for all $i = 0, 1, \ldots, d$. Then there exists an operator $X_P^L \succ 0$ satisfying (12) if and only if, for all $k \geq 0$, $i = 0, 1, \ldots, d$,*

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} \succeq 0 \quad \text{and} \quad \mathrm{rank} \begin{bmatrix} Y_i(k) & I \\ I & X_i(k) \end{bmatrix} \leq n_i(k) + m_i(k).$$

The proof of this is nearly identical to its matrix version found in Packard (1994) and so we do not include it here.

The next theorem transforms (11) into convex conditions.

Theorem 8. *There exists an admissible synthesis K_δ for NSLPV plant $\mathcal{G}_\delta$, with dimensions $m_i \leq n_i$ for all $i = 0, 1, \ldots, d$, if there exist operators $X, Y \in \mathcal{X}$ such that*

$$\begin{aligned} &N_Y^* \left\{ F \begin{bmatrix} Y & \\ & I \end{bmatrix} F^* - \begin{bmatrix} \widetilde{Z}^* Y \widetilde{Z} & \\ & I \end{bmatrix} \right\} N_Y \prec 0, \\ &N_X^* \left\{ F^* \begin{bmatrix} \widetilde{Z}^* X \widetilde{Z} & \\ & I \end{bmatrix} F - \begin{bmatrix} X & \\ & I \end{bmatrix} \right\} N_X \prec 0, \\ &\begin{bmatrix} Y & I \\ I & X \end{bmatrix} \succeq 0, \quad \text{with } F = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \end{aligned} \tag{13}$$

where the operators N_Y, N_X satisfy

$$\mathrm{Im}\, N_Y = \mathrm{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}, \qquad N_Y^* N_Y = I,$$
$$\mathrm{Im}\, N_X = \mathrm{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \qquad N_X^* N_X = I.$$

Proof. Suppose there exist operators X and Y satisfying the conditions in (13). Then by invoking Lemma 7, for some positive integers $m_i \leq n_i$, there exists an operator X_P^L that satisfies (12). Set $N_Y^* = \begin{bmatrix} V_1^* & V_2^* \end{bmatrix}$, where V_2 is block diagonal with $V_2(k) \in \mathbb{R}^{n_z(k)\times ?}$, and V_1 is a partitioned operator, each of whose elements is block diagonal, namely $V_1 = \begin{bmatrix} V_1^{0*} & V_1^{1*} & \cdots & V_1^{d*} \end{bmatrix}^*$ with $V_1^i(k) \in \mathbb{R}^{n_i(k)\times ?}$. Then, the first condition in the theorem statement is equivalent to

$$\begin{bmatrix} H & V_1^* B_1 + V_2^* D_{11} \\ B_1^* V_1 + D_{11}^* V_2 & -I \end{bmatrix} \prec 0,$$

where $H = V_1^*(AYA^* - \widetilde{Z}^* Y \widetilde{Z})V_1 + V_1^* AYC_1^* V_2 + V_2^* C_1 YA^* V_1 + V_2^*(C_1 Y C_1^* - I)V_2$. Applying the Schur complement formula to this condition, we equivalently get

$$-V_1^* \widetilde{Z}^* Y \widetilde{Z} V_1 - V_2^* V_2 + \begin{bmatrix} V_1^* A + V_2^* C_1 & 0 & V_1^* B_1 + V_2^* D_{11} \end{bmatrix} \times \begin{bmatrix} Y & Y^b & 0 \\ (Y^b)^* & Y^c & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^* V_1 + C_1^* V_2 \\ 0 \\ B_1^* V_1 + D_{11}^* V_2 \end{bmatrix} \prec 0.$$

In order to invert $(X_P^L)^{-1} = \begin{bmatrix} Y & Y^b \\ (Y^b)^* & Y^c \end{bmatrix}$, we apply the Schur complement formula again and so the last inequality holds if and only if the following operator

$$\begin{bmatrix} -V_1^* \widetilde{Z}^* Y \widetilde{Z} V_1 - V_2^* V_2 & V_1^* A + V_2^* C_1 & 0 & V_1^* B_1 + V_2^* D_{11} \\ A^* V_1 + C_1^* V_2 & -X & -X^b & 0 \\ 0 & -(X^b)^* & -X^c & 0 \\ B_1^* V_1 + D_{11}^* V_2 & 0 & 0 & -I \end{bmatrix}$$

is negative definite. Setting $W_R = \begin{bmatrix} V_1^* & 0 & 0 & V_2^* \\ 0 & 0 & I_{\ell_2}^{(\bar{n},\bar{m},n_w)} & 0 \end{bmatrix}^*$, it is not difficult to see that the preceding negative definite operator is exactly $W_R^* H_{X_P^L} W_R$. Observe that, given R from (10), $\operatorname{Im} W_R = \operatorname{Ker} R$ and $W_R^* W_R = I$. A similar argument starting with the second condition in the theorem statement shows that the condition $W_Q^* H_{X_P^L} W_Q \prec 0$ from (11) holds. Thus, we have shown that conditions (13) hold if and only if there exists an X_P^L satisfying (12) such that $W_R^* H_{X_P^L} W_R \prec 0$ and $W_Q^* H_{X_P^L} W_Q \prec 0$. Therefore, by Lemma 6, there exists an admissible synthesis for $\mathcal{G}_\delta$. ■

Note that if we partition Y and X as: $Y = \operatorname{diag}(Y_s, Y_p)$ and $X = \operatorname{diag}(X_s, X_p)$, where $Y_s = Y_0$, $Y_p = \operatorname{diag}(Y_1, \ldots, Y_d)$, with X_s and X_p defined similarly, and if we define

$$\operatorname{Im} [\![N_Y]\!]_k = \operatorname{Ker} \begin{bmatrix} [\![B_2^*]\!]_k & D_{12}^*(k) \end{bmatrix},$$
$$\operatorname{Im} [\![N_X]\!]_k = \operatorname{Ker} \begin{bmatrix} [\![C_2]\!]_k & D_{21}(k) \end{bmatrix},$$

which are directly related to N_Y and N_X, then the conditions of Theorem 8 are clearly equivalent to the existence of $\beta > 0$ such that, for all $k \geq 0$, we have

(S1) $[\![N_Y]\!]_k^* \left\{ [\![F]\!]_k \operatorname{diag}\left(Y_s(k), [\![Y_p]\!]_k, I\right) [\![F]\!]_k^* - \operatorname{diag}\left(Y_s(k+1), [\![Y_p]\!]_k, I\right) \right\} [\![N_Y]\!]_k \prec -\beta I,$

(S2) $[\![N_X]\!]_k^* \left\{ [\![F]\!]_k^* \operatorname{diag}\left(X_s(k+1), [\![X_p]\!]_k, I\right) [\![F]\!]_k - \operatorname{diag}\left(X_s(k), [\![X_p]\!]_k, I\right) \right\} [\![N_X]\!]_k \prec -\beta I,$

(S3) $\begin{bmatrix} [\![Y]\!]_k & I \\ I & [\![X]\!]_k \end{bmatrix} \succeq 0, \quad \text{where } F = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}.$

This gives a recursive matrix form of the solution.

We now briefly outline the procedure for constructing a controller from the solutions X and Y. To start, define $m_i(k) := \operatorname{rank}([\![X_i]\!]_k - [\![Y_i]\!]_k) \leq n_i(k)$ for $i = 0, 1, \ldots, d$. Using Lemma 7, construct an operator X_P^L satisfying (12). Then, solve inequality (10) for the controller realization J. All of the preceding steps are convex but infinite-dimensional computations. See Gahinet and Apkarian (1991) or Packard (1994) for more details on this procedure.

5. Eventually periodic systems

We start by defining an eventually periodic operator.

Definition 9. A block-diagonal mapping P on ℓ_2 is (h, q)-eventually periodic if, for some integers $h \geq 0$, $q \geq 1$, we have $Z^q((Z^*)^h P Z^h) = ((Z^*)^h P Z^h) Z^q$, that is P is q-periodic after an initial transient behaviour up to time h. Moreover, a partitioned operator, whose elements are block diagonal, is (h, q)-eventually periodic if each of its block-diagonal elements is (h, q)-eventually periodic.

Note that if $h = 0$, then P is simply q-periodic. An NSLPV system is (h, q)-eventually periodic if each of its state space system operators is (h, q)-eventually periodic.

Proposition 10. *The following hold:*

(i) *If A^L, B^L, C^L, and D^L are (h, q)-eventually periodic, then there exists a solution in $\mathcal{X}^L$ to inequality (8) if and only if there exists an (N, q)-eventually periodic solution in $\mathcal{X}^L$ for some integer $N \geq h$;*

(ii) *If the NSLPV plant $\mathcal{G}_\delta$ is (h, q)-eventually periodic, then there exist solutions in $\mathcal{X}$ to inequalities (13) if and only if there exist (N, q)-eventually periodic solutions in $\mathcal{X}$ for some integer $N \geq h$.*

The proofs of Parts (i) and (ii) are very similar to those of Farhood and Dullerud (2002, Lemma 7) and Farhood and Dullerud (2005, Theorem 8), respectively. Note that, in the standard LTV case, the finite horizon length N in Part (i) can be chosen equal to h, as shown in Farhood and Dullerud (2002, 2005). This however is not necessarily true in the NSLPV case, and it is not difficult to construct counter examples to verify this. Moreover, even in the standard LTV case, N is not necessarily equal to h in Part (ii). In the case of q-periodic operators and systems ($h = 0$), N in both parts can be chosen equal to zero; this follows by a similar averaging technique to that used in the proof of Dullerud and Lall (1999, Theorem 20).

So, given an eventually periodic LPV system, it follows from Lemma 3 and Propositions 4 and 10 that inequality (8) reduces to a finite dimensional convex condition for determining an upper bound on the ℓ_2-induced norm of the system. The next result stems from Theorem 8 and Proposition 10.

Corollary 11. *Suppose the NSLPV plant $\mathcal{G}_\delta$ is (h, q)-eventually periodic. Then, for some integer $N \geq h$, there exists an admissible (N, q)-eventually periodic synthesis K_δ for $\mathcal{G}_\delta$, with dimensions $m_i \leq n_i$ for all $i = 0, 1, \ldots, d$, if there exist*

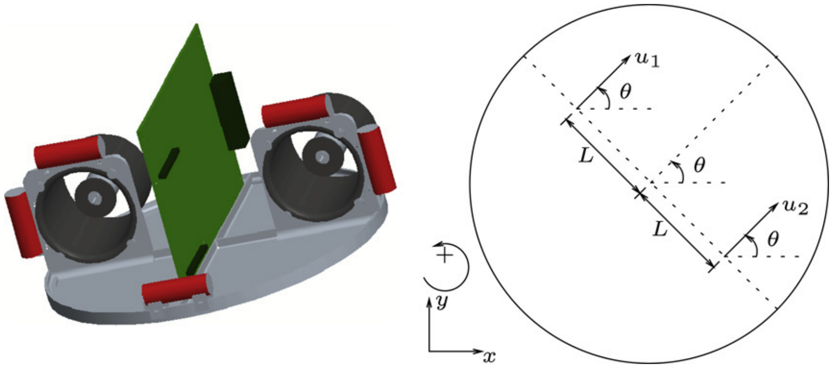

Fig. 1. Two Thruster Hovercraft.

positive definite solutions satisfying the synthesis conditions (S1–S3) for all $k = 0, 1, \ldots, N + q - 1$, *with* $X_s(N + q) = X_s(N)$ *and* $Y_s(N + q) = Y_s(N)$.

Solutions X and Y can be used to construct an (N, q)-eventually periodic controller K_δ. We remark that it may be possible to construct from these solutions an (M, q)-eventually periodic controller, where $h \le M \le N$, as suggested by Part (i) of Proposition 10. Note that generally we seek a γ-admissible synthesis, namely one that guarantees closed-loop stability as well as the norm condition $\|w \mapsto z\| < \gamma$ for some γ. Clearly, a γ-admissible synthesis for $\mathcal{G}_\delta$ is a 1-admissible synthesis for $\bar{\mathcal{G}}_\delta$, where $\bar{\mathcal{G}}_\delta$ has the same realization as $\mathcal{G}_\delta$ except that $\bar{C}_{1s} = \frac{1}{\gamma} C_{1s}$, $\bar{C}_{1p} = \frac{1}{\gamma} C_{1p}$, $\bar{D}_{11} = \frac{1}{\gamma} D_{11}$, and $\bar{D}_{12} = \frac{1}{\gamma} D_{12}$, and so, the previous synthesis results are still applicable in this case. Furthermore, by employing the Schur complement formula, it is possible to ensure that the synthesis conditions are also linear in γ (or γ^2 as in Farhood and Dullerud (2005, Problem (16))) and hence transform the synthesis feasibility problem into a convex optimization one to find the minimum γ.

6. Example: Control of a two-thruster hovercraft

We now apply the NSLPV approach to control a two-thruster hovercraft along an eventually periodic trajectory. This hovercraft, shown in Fig. 1, floats on an air cushion caused by a continuous air flow through a perforated sheet underneath it. This causes the hovercraft to glide on an almost frictionless surface. The two thrusters are positioned equidistantly from the central axis of the craft, where $L = 0.15$ m. These thrusters can only push in the forward direction, and each can exert a force of at most 2.5 Newtons. Following are the translational and rotational data for this system: mass $m = 1.731$ kg, translational friction $b_t = 3.7 \times 10^{-3}$ N s/m, rotational friction $b_r = 3.65 \times 10^{-4}$ N s m/rad, and polar moment of inertia $\mathcal{I} = 2.36328 \times 10^{-2}$kg m^2.

6.1. Nonlinear model and reference trajectory

Appealing to the free-body diagram in Fig. 1, the equations of motion for the hovercraft are given by

$$m\ddot{x} + b_t \dot{x} = (u_1 + u_2)\cos\theta, \tag{14}$$

$$m\ddot{y} + b_t \dot{y} = (u_1 + u_2)\sin\theta, \tag{15}$$

$$\mathcal{I}\ddot{\theta} + b_r \dot{\theta} = (u_2 - u_1)L. \tag{16}$$

The next step is to design a feasible eventually periodic reference trajectory $(x_r(t), y_r(t), \theta_r(t))$, and a corresponding reference input $u_r(t) = (u_{1r}(t), u_{2r}(t))$ achievable by the thrusters. We assume that the control input is applied in discrete-time with a sampling frequency of 20 Hz; namely for all integers $k \ge 0$ and $i = 1, 2$, we have: $u_{ir}(t) = u_{ir,k} \in [0, 2.5]$ for all $kT \le t < (k + 1)T$, where $T = 0.05$ s is the sampling period. We choose the reference path to be of an elliptical nature: the finite horizon path is part of an ellipse of semimajor axis equal to 1 and semiminor axis equal to 0.825, and the periodic path is also an ellipse of semimajor and semiminor axes equal to 1 and 0.75 respectively. We can then easily parameterize the reference coordinates in terms of the directed angle ϕ, shown in Fig. 2. To ensure that the reference input is achievable by the thrusters, we choose $\phi(t)$ such that the angular speed $\dot{\phi}$ is constant and of reasonable magnitude ($\dot{\phi} = \frac{\pi}{3}$ rad/s) over the periodic part. As for the finite horizon part, since we assume the hovercraft is initially at rest, we choose $\phi(t)$ to be a polynomial of third degree which connects to its periodic counterpart at a point where $\dot{\phi} = \frac{\pi}{3}$ and $\ddot{\phi} = 0$; this ensures a smooth transition between the finite horizon and the periodic part. Specifically, $x_r(t)$ and $y_r(t)$ are defined as follows:

$$\begin{array}{ll}\text{Finite Horizon} & \left\{\begin{array}{l} x_r = 0.825\cos\phi\left(1 - 0.319375\cos^2\phi\right)^{-\frac{1}{2}} \\ y_r = 0.825\sin\phi\left(1 - 0.319375\cos^2\phi\right)^{-\frac{1}{2}} \\ \phi = -\dfrac{16\pi}{729}t^3 + \dfrac{4\pi}{27}t^2 + \dfrac{\pi}{2} \end{array}\right. \\ t \in [0, 2.25) & \\ \text{Periodic Part} & \left\{\begin{array}{l} x_r = 0.75\cos\phi\left(1 - 0.4375\cos^2\phi\right)^{-\frac{1}{2}} \\ y_r = 0.75\sin\phi\left(1 - 0.4375\cos^2\phi\right)^{-\frac{1}{2}} \\ \phi = \dfrac{\pi}{3}(t - 2.25) + \pi. \end{array}\right. \\ t \in [2.25, 8.25) & \end{array}$$

In the sequel, given a function $p(t)$, we use p_k to denote the value of $p(t)$ at time kT. We now solve Eqs. (14) and (15) for θ_r and $u_{1r} + u_{2r}$. As $x_r(t)$ and $y_r(t)$ are eventually periodic, we only need to account for the finite-horizon first-period truncations of these functions; clearly, the resulting functions $\theta_r(t)$ and $u_r(t)$ will be eventually periodic as well. Since T is sufficiently small, we may assume for simplicity that θ_r

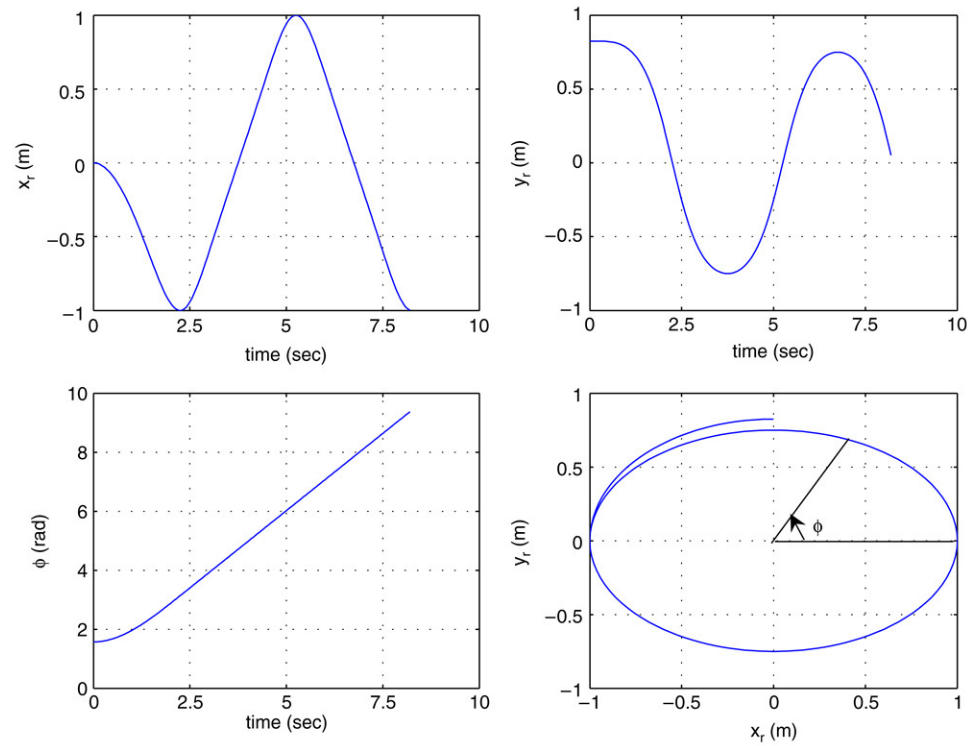

Fig. 2. Reference eventually periodic path.

varies linearly in time over each interval $[kT, (k+1)T]$ for $k = 0, 1, \ldots, 164$, i.e. for $kT \leq t \leq (k+1)T$, we have $\theta_r(t) = a_k \cdot (t - kT) + \theta_{r,k}$, where a_k is a constant. Then, as u_r is constant in each interval $(kT, (k+1)T)$, integrating (14) and (15) over these intervals gives $a_k = \frac{2}{T}\left(\operatorname{arccot} E - \theta_{r,k}\right) + \frac{2\epsilon\pi}{T}$,

$$u_{1r,k} + u_{2r,k} = a_k \frac{m\left(\dot{y}_{r,k+1} - \dot{y}_{r,k}\right) + b_t\left(y_{r,k+1} - y_{r,k}\right)}{\cos\theta_{r,k} - \cos\theta_{r,k+1}},$$

where $E = \dfrac{m\left(\dot{x}_{r,k+1} - \dot{x}_{r,k}\right) + b_t\left(x_{r,k+1} - x_{r,k}\right)}{m\left(\dot{y}_{r,k+1} - \dot{y}_{r,k}\right) + b_t\left(y_{r,k+1} - y_{r,k}\right)}$,

$\theta_{r,0} = -\pi$, and ϵ is an integer chosen so that $|a_k|$ is the minimum possible. Clearly, the function $\theta_r(t)$, as given above, is not differentiable, and hence we cannot solve Eq. (16). To fix this problem, we fit the data $\theta_{r,k}$ to a smoothing spline using the matlab command *fit*, and then differentiate the resulting spline. Afterwards, integrating Eq. (16) over each of the time intervals $(kT, (k+1)T)$ for $k = 0, 1, \ldots, 164$, we can solve for $u_{2r} - u_{1r}$, and consequently u_r, shown in Fig. 3. With the preceding approximations in mind, solving the differential Eqs. (14) and (16), given this reference input, would generate the reference trajectory almost exactly over the finite horizon and the first couple of periods, but, afterwards, the accumulating approximation errors will slowly become significant and the resulting trajectory would start deviating from the reference one. This, however, is inconsequential in our case as we ultimately seek to design a robust feedback controller.

6.2. NSLPV synthesis

First, we derive an NSLPV model. Define the state column vector $v = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$, the input column vector $u = (u_1, u_2)$, and the associated errors $\bar{v} = v - v_r$ and $\bar{u} = u - u_r$. The Eqs. (14) and (16) can be equivalently written as $\dot{v} = f(v, u)$, where $f(\cdot,\cdot)$ is defined in the obvious way. We have $\dot{\bar{v}} = \dot{v} - \dot{v}_r = f(v,u) - f(v_r, u_r)$; it is easy to see that the nonlinear terms in this equation are $P_1 = \frac{1}{m}\sum_{i=1}^{2}(u_i\cos\theta - u_{ir}\cos\theta_r)$ and $P_2 = \frac{1}{m}\sum_{i=1}^{2}(u_i\sin\theta - u_{ir}\sin\theta_r)$. Then one choice for parametrization is $\delta = \bar{\theta} = \theta - \theta_r$. As we seek rational dependence on the parameters, we have to approximate the cosine and sine functions with Taylor polynomials, namely

$$\begin{aligned}
\cos\theta &= \cos(\bar{\theta} + \theta_r) \approx \sum_{i=0}^{5} a_i \bar{\theta}^i \\
&= \cos\theta_r - (\sin\theta_r)\bar{\theta} - \frac{\cos\theta_r}{2!}\bar{\theta}^2 + \frac{\sin\theta_r}{3!}\bar{\theta}^3 \\
&\quad + \frac{\cos\theta_r}{4!}\bar{\theta}^4 - \frac{\sin\theta_r}{5!}\bar{\theta}^5, \\
\sin\theta &= \sin(\bar{\theta} + \theta_r) \approx \sum_{i=0}^{5} c_i \bar{\theta}^i \\
&= \sin\theta_r + (\cos\theta_r)\bar{\theta} - \frac{\sin\theta_r}{2!}\bar{\theta}^2 - \frac{\cos\theta_r}{3!}\bar{\theta}^3 \\
&\quad + \frac{\sin\theta_r}{4!}\bar{\theta}^4 + \frac{\cos\theta_r}{5!}\bar{\theta}^5.
\end{aligned}$$

Instead of truncating the Taylor series expansions for these trigonometric functions, we may alternatively assign parameters to the remainder terms as given by Taylor's theorem, and then use the Remainder Estimation Theorem to get bounds on these additional parameters. In this scenario, the terms $\sin\theta$ and $\cos\theta$ would be equivalently written as polynomial functions of the parameters. However, for all our purposes here,

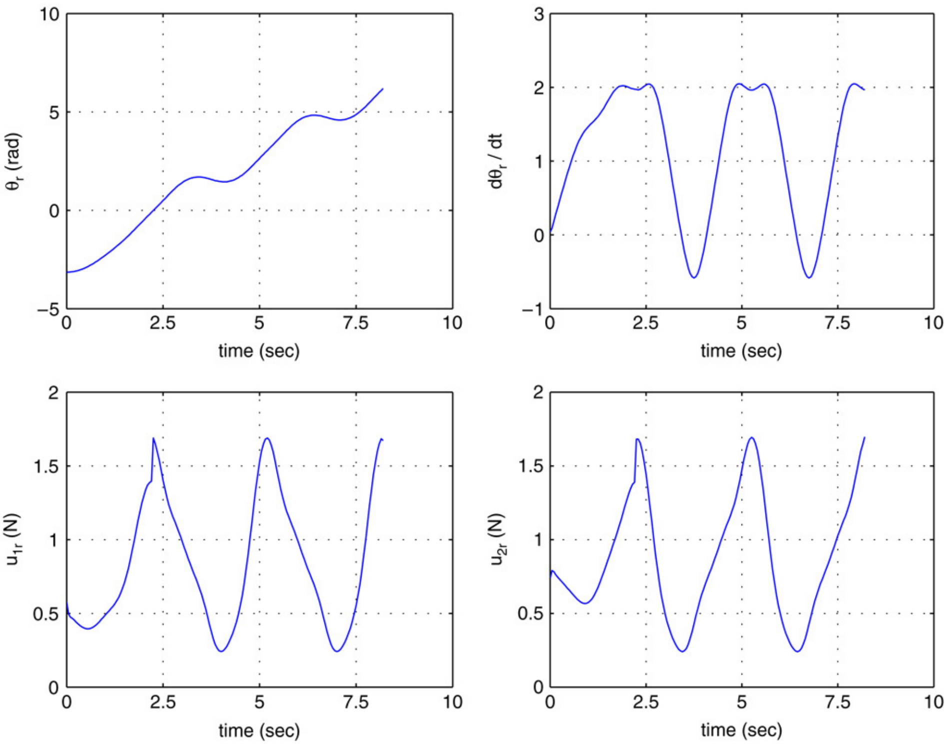

Fig. 3. Reference angle θ and control input.

such a parametrization needlessly complicates the resulting NSLPV model. Now, some algebra leads to

$$P_1 \approx \psi_1(\bar{\theta}, t)\bar{\theta} + \begin{bmatrix}\psi_2(\bar{\theta}, t) & \psi_2(\bar{\theta}, t)\end{bmatrix}\bar{u},$$
$$P_2 \approx \rho_1(\bar{\theta}, t)\bar{\theta} + \begin{bmatrix}\rho_2(\bar{\theta}, t) & \rho_2(\bar{\theta}, t)\end{bmatrix}\bar{u},$$

where $\psi_1(\bar{\theta}, t) = \sum_{j=0}^{4} \kappa_j \ \bar{\theta}^j$, $\psi_2(\bar{\theta}, t) = \sum_{j=0}^{5} \lambda_j \ \bar{\theta}^j$, $\rho_1(\bar{\theta}, t) = \sum_{j=0}^{4} \mu_j \ \bar{\theta}^j$, $\rho_2(\bar{\theta}, t) = \sum_{j=0}^{5} \xi_j \ \bar{\theta}^j$, $\lambda_j = \frac{a_j}{m}$, $\kappa_j = \frac{u_{1r}+u_{2r}}{m} a_{j+1}$, $\xi_j = \frac{c_j}{m}$, and $\mu_j = \frac{u_{1r}+u_{2r}}{m} c_{j+1}$.

As a result, we get the continuous-time state-space equation:

$$\dot{\bar{v}} = \mathcal{A}(\bar{\theta}, t)\bar{v} + \mathcal{B}(\bar{\theta}, t)\bar{u}, \tag{17}$$

where $\mathcal{A}(\bar{\theta}, t)$ and $\mathcal{B}(\bar{\theta}, t)$ are equal to

$$\begin{bmatrix} 0_{3\times 3} & I_3 \\ \begin{bmatrix} 0 & 0 & \psi_1(\bar{\theta}, t) \\ 0 & 0 & \rho_1(\bar{\theta}, t) \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -\frac{b_t}{m} & 0 & 0 \\ 0 & -\frac{b_t}{m} & 0 \\ 0 & 0 & -\frac{b_r}{\mathcal{I}} \end{bmatrix} \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} 0_{3\times 2} \\ \begin{bmatrix} \psi_2(\bar{\theta}, t) & \psi_2(\bar{\theta}, t) \\ \rho_2(\bar{\theta}, t) & \rho_2(\bar{\theta}, t) \\ -\frac{L}{\mathcal{I}} & \frac{L}{\mathcal{I}} \end{bmatrix} \end{bmatrix},$$

respectively. Next, we formulate this equation in an LFT framework. We will find the following notation convenient:

$$\delta I \star \mathcal{M} = \mathcal{M}_{21}(I - \delta\mathcal{M}_{11})^{-1}\delta\mathcal{M}_{12} + \mathcal{M}_{22},$$

where $\mathcal{M} = \left[\begin{array}{c:c} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \hdashline \mathcal{M}_{21} & \mathcal{M}_{22} \end{array}\right]$.

Our goal is to equivalently present state-space equation (17) in the following LFT format:

$$\dot{\bar{v}} = A_{ss}^c(t)\bar{v} + A_{sp}^c(t)\beta_c + B_{2s}^c(t)\bar{u},$$
$$\alpha_c = A_{ps}^c(t)\bar{v} + A_{pp}^c(t)\beta_c + B_{2p}^c(t)\bar{u}, \qquad \beta_c = \bar{\theta}\alpha_c.$$

In other words, we need to write the matrix-valued functions $\mathcal{A}(\bar{\theta}, t)$ and $\mathcal{B}(\bar{\theta}, t)$ as

$$\bar{\theta} I \star \left[\begin{array}{c:c} A_{pp}^c(t) & A_{ps}^c(t) \\ \hdashline A_{sp}^c(t) & A_{ss}^c(t) \end{array}\right] \quad \text{and} \quad \bar{\theta} I \star \left[\begin{array}{c:c} A_{pp}^c(t) & B_{2p}^c(t) \\ \hdashline A_{sp}^c(t) & B_{2s}^c(t) \end{array}\right], \tag{18}$$

respectively. It is not difficult to see that

$$\mathcal{A}(\bar{\theta}, t) = \bar{\theta} I_4 \star \left[\begin{array}{c:c} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \hdashline \mathcal{A}_{21} & \mathcal{A}_{22} \end{array}\right] = \bar{\theta} I_4 \star \left[\begin{array}{c:c} 0_{1\times 4} & E_0 \\ \begin{bmatrix} I_3 & 0_{3\times 1} \end{bmatrix} & 0_{3\times 6} \\ \hdashline 0_{3\times 4} & \begin{bmatrix} 0_{3\times 3} & I_3 \end{bmatrix} \\ E_1 & E_2 \end{array}\right]$$

and $$\mathcal{B}(\bar{\theta}, t) = \bar{\theta} I_5 \star \left[\begin{array}{c:c} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \hdashline \mathcal{B}_{21} & \mathcal{B}_{22} \end{array}\right] = \bar{\theta} I_5 \star \left[\begin{array}{c:c} 0_{1\times 5} & F_0 \\ \begin{bmatrix} I_4 & 0_{4\times 1} \end{bmatrix} & 0_{4\times 2} \\ \hdashline 0_{3\times 5} & 0_{3\times 2} \\ F_1 & F_2 \end{array}\right],$$

where $E_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$, $F_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}$,

$$E_1 = \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & \kappa_0 & -\frac{b_t}{m} & 0 & 0 \\ 0 & 0 & \mu_0 & 0 & -\frac{b_t}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{b_r}{\mathcal{I}} \end{bmatrix},$$

$$F_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} \lambda_0 & \lambda_0 \\ \xi_0 & \xi_0 \\ -\frac{L}{\mathcal{I}} & \frac{L}{\mathcal{I}} \end{bmatrix}.$$

It is obvious from the preceding that the matrix-valued system functions in (18) can be chosen as follows:

$$A^c_{pp} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ 0 & \mathcal{B}_{11} \end{bmatrix}, \qquad A^c_{ps} = \begin{bmatrix} \mathcal{A}_{12} \\ 0 \end{bmatrix}, \qquad B^c_{2p} = \begin{bmatrix} 0 \\ \mathcal{B}_{12} \end{bmatrix},$$
$$A^c_{sp}(t) = \begin{bmatrix} \mathcal{A}_{21} & \mathcal{B}_{21} \end{bmatrix},$$
$$A^c_{ss}(t) = \mathcal{A}_{22}, \qquad B^c_{2s}(t) = \mathcal{B}_{22}.$$

In order to simplify the discretization of the continuous-time LFT model, and since the sampling period T is sufficiently small, it is reasonable to assume that the scheduled parameter δ varies very slowly in time interval $[kT, (k+1)T)$ that its values on this interval can be approximated by $\delta_k = \bar{\theta}(kT)$. Then, we can use zero-order hold sampling to obtain the following discrete-time state-space equation:

$$\bar{v}_{k+1} = A_{ss,k}\bar{v}_k + A_{sp,k}\beta_k + B_{1s,k}w_k + B_{2s,k}\bar{u}_k,$$

where $A_{ss,k} = \Phi_{ss}((k+1)T, kT)$, Φ_{ss} being the state transition matrix associated with $A^c_{ss}(t)$, $\bar{v}_k = \bar{v}(kT)$, $\beta_k = \beta_c(kT)$, $A_{sp,k} = \int_{kT}^{(k+1)T} \Phi_{ss}((k+1)T, \tau)\, A^c_{sp}(\tau)\mathrm{d}\tau$, $B_{is,k} = \int_{kT}^{(k+1)T} \Phi_{ss}((k+1)T, \tau)\, B^c_{is}(\tau)\mathrm{d}\tau$ for $i = 1, 2$, with $B^c_{1s} = \begin{bmatrix} 0_{3\times3} & I_3 \end{bmatrix}^*$ (i.e. the disturbances w are in the form of torques as well as forces in the x and y directions, applied like the input in discrete time with a sampling frequency of 20 Hz). Alternatively, as proposed in Apkarian (1997), we can use a bilinear transformation to obtain a discrete-time trapezoidal approximation.

We assume that the parameter $\delta = \bar{\theta}$ is such that $|\delta| \le \frac{\pi}{6}$. Then, this bound is absorbed into the plant so that the new scaled parameter $\bar{\delta}$ satisfies $|\bar{\delta}| \le 1$, where $\delta = \frac{\pi}{6}\bar{\delta}$. Also, due to this scaling, we get $A_{ps} = \frac{\pi}{6}A^c_{ps}$, $A_{pp} = \frac{\pi}{6}A^c_{pp}$, $B_{1p} = \frac{\pi}{6}B^c_{1p}$, $B_{2p} = \frac{\pi}{6}B^c_{2p}$. We assume that the states x, y, and θ are exactly measurable, and as for the exogenous errors to be controlled, we choose to equally penalize $\bar{x}$, $\bar{y}$, $\bar{\theta}$, $\bar{u}_1$, and $\bar{u}_2$. Then, we get the discrete-time (45, 120)-eventually periodic LPV model: $\beta_k = \bar{\delta}_k\alpha_k,\ |\bar{\delta}_k| \le 1$,

$$\begin{bmatrix} \bar{v}_{k+1} \\ \alpha_k \\ z_k \\ p_k \end{bmatrix} = \begin{bmatrix} A_{ss,k} & A_{sp,k} & B_{1s,k} & B_{2s,k} \\ A_{ps} & A_{pp} & B_{1p} & B_{2p} \\ C_{1s} & C_{1p} & D_{11} & D_{12} \\ C_{2s} & C_{2p} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \bar{v}_k \\ \beta_k \\ w_k \\ \bar{u}_k \end{bmatrix}, \tag{19}$$

where $w \in \ell_2$, $\bar{v}(0) = 0$, B_{1p}, C_{1p}, C_{2p}, D_{11}, D_{21}, D_{22} are all zero matrices, and $C_{1s} = \mathrm{diag}(I_3, 0_{2\times3})$,

$$C_{2s} = \begin{bmatrix} I_3 & 0_{3\times3} \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 0_{2\times3} & I_2 \end{bmatrix}^*.$$

The LFT formulation has significantly increased the model dimensions; we now have six states as well as nine copies of the parameter $\bar{\delta}$. Clearly, a model reduction theory for such LFT models is important, and this is treated in-depth in Farhood and Dullerud (2007). In this case, however, since the minimality theory for transfer functions in a single complex variable is identical to that for rational functions in a single real variable, we can reduce the model dimensions pointwise in time at no cost. Specifically, appealing to (19), we have

$$\bar{v}_{k+1} = \underbrace{\left(\bar{\delta}_k I_9 \star \left[\begin{array}{c|ccc} A_{pp} & A_{ps} & B_{1p} & B_{2p} \\ \hline A_{sp,k} & A_{ss,k} & B_{1s,k} & B_{2s,k} \end{array}\right]\right)}_{\mathcal{H}(\bar{\delta}_k)} \begin{bmatrix} \bar{v}_k \\ w_k \\ \bar{u}_k \end{bmatrix},$$

and so, at each k, we can reduce the dimensions of the model by eliminating any uncontrollable or unobservable states of the real variable "transfer function" $\mathcal{H}(\bar{\delta}_k)$, and hence obtaining the minimal realization of $\mathcal{H}(\bar{\delta}_k)$. Doing so, we end up with a reduced LFT model with six states and five copies of $\bar{\delta}$ at each time instant.

Appealing to Corollary 11 and its subsequent discussion, we can solve for a $\gamma_{\min}$-admissible (45, 120)-eventually periodic LPV synthesis K_δ, where $\gamma_{\min}$ is the minimum achievable γ by such a synthesis. Using SeDuMi (Sturm, 1999), we find that, in this case, $\gamma_{\min} \approx 2.57$, whereas in the counterpart LTV case (i.e. no parameters) its value would be about 2. Clearly, as the bound on the parameter $\bar{\theta}$ decreases, the value of $\gamma_{\min}$ potentially decreases too, but it may not in general converge to the corresponding LTV value.

6.3. Simulation

Note that it is not possible to tackle this control problem using a stationary approach; this is mainly due to the restrictions on the control input. Specifically, it is not difficult to show via counter examples that the LTI model, obtained from linearizing the nonlinear system equations about any stationary point, is not asymptotically stabilizable by nonnegative control. Hence, a nonstationary approach is necessary in this case. The NSLPV controller turns out to be quite robust in the presence of significant model uncertainties even though this is not deliberate by design. This robustness is probably due to the NSLPV model formulation and the fact that the NSLPV controller is designed to work for all permissible parameter trajectories. To elaborate, suppose we increase the mass and inertia of the hovercraft by 50%, and subject this vehicle to iid disturbances, generated by the matlab function *rand*; these disturbances correspond to forces in the x and y directions, namely $\mathcal{F}_{xk}$ and $\mathcal{F}_{yk}$, as well as torques $\mathcal{T}_k$, applied on the hovercraft in discrete-time with a sampling frequency of 20 Hz, such that $|\mathcal{F}_{xk}|, |\mathcal{F}_{yk}| \le 1$ N and $|\mathcal{T}_k| \le 0.075$ N m for all integers $k \ge 0$. We find that, despite these disturbances and significant model uncertainties, the NSLPV controller still manages to force the hovercraft to track the trajectory rather closely, as shown in Fig. 4. Movies of this simulation and others can be found at http://legend.me.uiuc.edu/~mazen/NSLPVcontrol/.

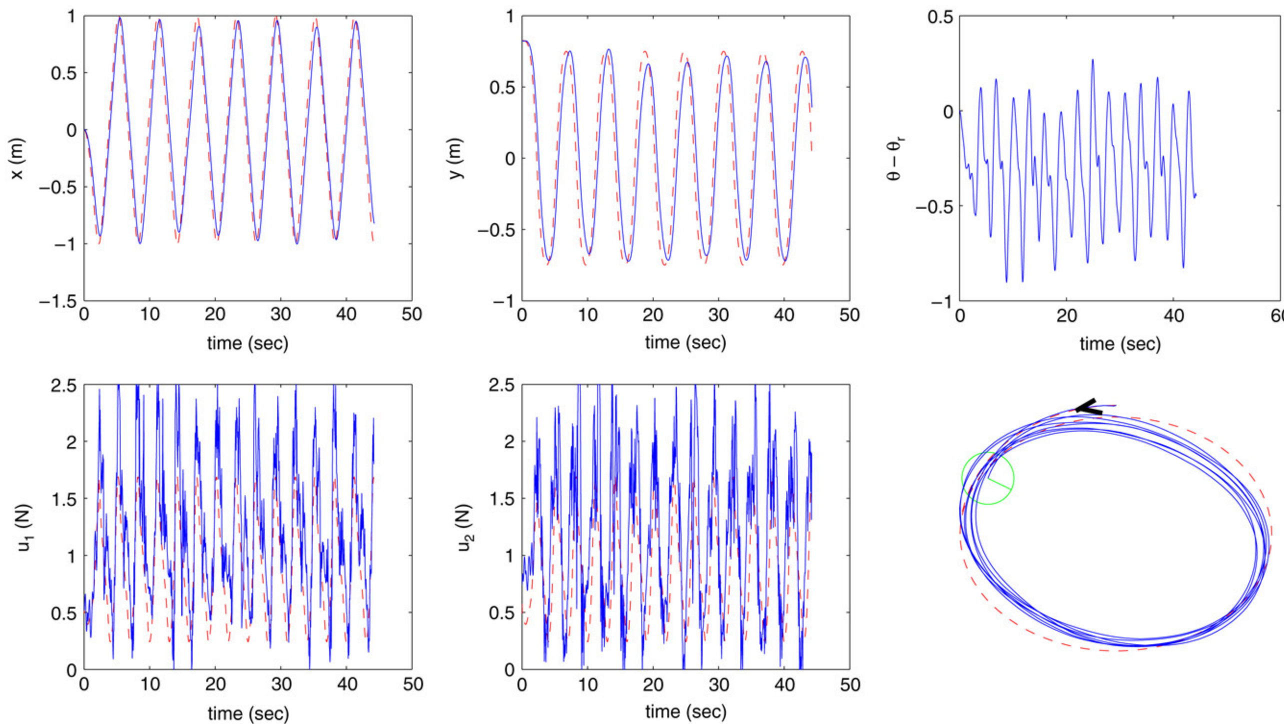

Fig. 4. NSLPV simulation (dashed curves correspond to reference).

7. Conclusions

This paper gives results for the control of nonstationary LPV systems, which are analogous to those for stationary LPV systems. The motivation for this work is a systematic method for gain scheduling of systems controlled along *prespecified* trajectories. In this context a benefit of using a nonstationary model is to reduce the conservatism introduced when capturing the behaviour of a nonlinear system in an LPV model.

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